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Flux-vacua in two dimensional string theory

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Abstract: We analyze the two dimensional type 0 theory with background RR-fluxes. Both the 0A and the 0B theory have two distinct fluxes q and $\widetilde{q}$. We study these two theories at finite temperature (compactified on a euclidean circle of radius R) as a function of the fluxes, the tachyon condensate μ and the radius R. Surprisingly, the dependence on q, $\widetilde{q}$ and μ is rather simple. The partition function is the absolute value square of a holomorphic function of $y = |q| + |\widetilde{q}| + i\sqrt{2\alpha'}\mu$ (up to a simple but interesting correction). As expected, the 0A and the 0B answers are related by T-duality. Our answers are derived using the exact matrix models description of these systems and are interpreted in the low energy spacetime lagrangian.

Keywords: Matrix Models, 2D Gravity, Conformal Field Models in String Theory.

Contents

1. Introduction

The renewed interest in noncritical string theories has originated from their relevance to current topics in string theory [1−3]-[4], like open/closed duality, holography and D-branes. These models are interesting because they have a complete nonperturbative definition and at the same time can be analyzed exactly. As such, they are good laboratories for subtle nonperturbative questions. In particular, this is the only framework where flux vacua — backgrounds with RR-fluxes — can be analyzed exactly. Issues associated with such flux vacua have already been discussed both in $\widehat{c} = 1$ models [3−17]-[18] and in $\widehat{c} < 1$ models [19, 20]-[21]. However, some of the results of the $\widehat{c} = 1$ system appeared confusing and it has been suggested that the system with RR-flux is related to black holes. The purpose of this paper is to clarify some of these confusions.

The $\widehat{c} = 1$ model is a two dimensional string theory. The target space is parameterized by the time t and the spatial coordinate ϕ. The background is not translational invariant; the system has a linear dilaton which makes the string coupling space dependent

$$g_s(\phi) = e^{-\phi}\,. \tag{1.1}$$

The $\phi \to +\infty$ asymptotic region is characterized by weak coupling. Scattering experiments are performed by sending signals from this asymptotic region and detecting them as they return. More precisely, the scattering is to and from null infinities $\mathcal{J}^{\pm}$; the incoming modes are functions of $\phi + t$ and the outgoing modes are functions of $\phi - t$. It has been assumed that the system does not have another asymptotic region with $\phi \to -\infty$; i.e. there is no separate scattering to and from that region. Indeed, the strong coupling region has effectively finite volume (see, e.g. [21]).

In section 2 we discuss the spacetime picture of the two kinds of models we study, the 0B and the 0A theories. We review their spectra and the leading order terms in the spacetime effective lagrangian. We show that the 0B theory has two kinds of continuous RR-fluxes, ν and $\widetilde{\nu}$, and 0A theory has two kinds of quantized RR-fluxes, q and $\widetilde{q}$.

In addition to these two parameters we can also turn on a "tachyon" condensate $\langle T(\phi)\rangle = \mu e^{-\phi}$ and study the physics as a function of the real parameter μ. The analysis of [4] showed that the physics is smooth as a function of μ. Finally, we can also study the system with euclidean time which is compactified on a circle of radius R. This corresponds to studying the thermodynamics of the theory with finite temperature $\frac{1}{2\pi R}$.

In section 3 we study the exact 0A theory with its two RR-fluxes and derive an expression for its partition function $\mathcal{Z}_A(\mu, q, \widetilde{q}, R)$ as a function of all variables. We find that up to a simple (but interesting) term, the dependence on q and $\widetilde{q}$ is only through $\widehat{q} = |q| + |\widetilde{q}|$. Furthermore, up to the same simple term, the partition function factorizes as a holomorphic function of $y = \widehat{q} + i\sqrt{2\alpha'}\mu$ and its complex conjugate. We interpret the dependence on $\widehat{q} = |q| + |\widetilde{q}|$ as a result of the presence of $|q\widetilde{q}|$ fundamental strings in the system. This is reminiscent of the factorization involved in topological string computations, see e.g. [22] and references therein.

The presence of two distinct fluxes in the 0A theory, which couple differently to the tachyon, has led to speculations about the existence of extremal black hole solutions when the tachyon is not excited: $\mu = 0$. Indeed the lowest order in α' equations of motion predict such a solution [23]. Unfortunately it is not possible to trust the leading order equations in the two dimensional string theory. The fact that our matrix model results depend only on $\widehat{q} = |q| + |\widetilde{q}|$ shows that the physics is essentially the same as the physics with only one kind of flux that has been analyzed in [4], see also [24] . Such analysis does not show any indications of a black hole, namely there is no entropy and there is no classical absorption. So the matrix model, as analyzed in this paper is not consistent with an object that could be called a black hole.

In section 4 we study the exact 0B theory. We view the partition function of the noncompact lorentzian theory $\mathcal{Z}_B$ as a *transition amplitude* between the past and the future. The value of this amplitude is complex, but is simpler than expected. Its phase is given by the real part of $\Xi\left(-2i\sqrt{2\alpha'}(|\nu| + |\widetilde{\nu}| + \mu)\right) + \Xi\left(-2i\sqrt{2\alpha'}(|\nu| + |\widetilde{\nu}| - \mu)\right)$ for some function Ξ and ν, $\widetilde{\nu}$ are the lorentzian RR fluxes. The finite temperature version of the 0B theory has quantized fluxes $|q| = -2iR|\nu|$ and $|\widetilde{q}| = -2iR|\widetilde{\nu}|$. The expression for the finite temperature partition function is related to that of the 0A theory by the expected T-duality with the following change in the parameters

$$R_B = \frac{\alpha'}{R_A}\,, \qquad \mu_B = \frac{R_A}{\sqrt{2\alpha'}}\mu_A \tag{1.2}$$

with the same q and $\widetilde{q}$.

In appendix A we review and extend a simple formalism for describing these systems [25]. It allows us to simply compute the transition amplitudes both of the 0B and the 0A theory and to obtain new insights into the nature of the scattering.

2. Spacetime effective lagrangian

In this section we focus on the weak coupling end of the target space, $\phi \to +\infty$ and study the low energy field theory there. Since the string coupling is arbitrarily small, the dynamics is dominated by classical physics, and the leading approximation to the effective field theory is valid. In particular, the massless modes are reliably found by a weak coupling worldsheet analysis. Another simplification in this part of the target space is that a possible tachyon condensate

$$\langle T(\phi)\rangle = \mu e^{-\phi} \tag{2.1}$$

can be neglected there.

2.1 0B

We start by analyzing the 0B string theory. The spectrum consists of two massless scalars an NS-NS "tachyon" T and an RR scalar C.

It is clear from the worldsheet description that the theory has two discrete $\mathbf{Z}_2$ symmetries [4]:

1. The first symmetry acts in the worldsheet description as $(-1)^{F_L}$ where F_L is the target space fermion number of the worldsheet left movers. It acts on the spectrum as

$$\begin{aligned} T &\to T \\ C &\to -C \end{aligned} \tag{2.2}$$

Since it changes the sign of the RR scalar, it changes the charge of D-branes; we will can refer to it as charge conjugation.

2. A more subtle symmetry acts in the worldsheet description as $(-1)^{f_L}$ where f_L is the left moving worldsheet fermion number. It acts on the spectrum as

$$\begin{aligned} T &\to -T \\ C_L &\to C_L \\ C_R &\to -C_R\,. \end{aligned} \tag{2.3}$$

Here C_L and C_R are the *target space* left and right moving components of C. Hence the action of this symmetry on C is a duality transformation. By analogy with its higher dimensional counterpart, we will refer to it as S-duality.

The invariance under S-duality means that the scalar C is compact and its radius is the selfdual radius. Therefore, the asymptotic theory as $\phi \to +\infty$ has an $\mathrm{SU}(2) \times \mathrm{SU}(2)$ symmetry. This symmetry will be important below.

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Since T is odd under the S-duality symmetry, the kinetic term of C has to be of the form $\frac{1}{8\pi}f(T)(\partial C)^2$ with $f(-T) = \frac{1}{f(T)}$. More detailed worldsheet considerations show that $f(T) = e^{2T}$ and hence the kinetic term is

$$\mathcal{L}_{kinetic} = \frac{1}{8\pi}e^{2T}\left[(\partial_t C)^2 - (\partial_\phi C)^2\right]. \tag{2.4}$$

Therefore, the coupling of T to C breaks the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ symmetry to $\mathrm{U}(1) \times \mathrm{U}(1)$. In particular, the tachyon condensate $\langle T(\phi)\rangle = \mu e^{-\phi}$ breaks the S-duality symmetry; more precisely, the theory with μ is related by S-duality to the theory with $-\mu$.

Let us examine the equations of motion which arise from (2.4)

$$\partial_t(e^{2T}\partial_t C) - \partial_\phi(e^{2T}\partial_\phi C) = 0\,. \tag{2.5}$$

For $\phi \to +\infty$ we can neglect T in this expression and C is simply a free scalar at the selfdual radius. Of particular interest to us will be the zero momentum solutions of the equations of motion

$$\frac{C}{\sqrt{2}} \approx 2(\nu\phi + \widetilde{\nu}t) = \nu_{\mathrm{in}}(\phi + t) + \nu_{\mathrm{out}}(\phi - t) \tag{2.6}$$

where we used an approximate sign to remind us that this solution is valid only for $\phi \to +\infty$. The two integration constants ν and $\widetilde{\nu}$, or equivalently ν_{in} and ν_{out} are RR-fluxes. Both of them are odd under the charge conjugation symmetry (2.2) and transform under S-duality (2.3) as $\nu \leftrightarrow \widetilde{\nu}$, or equivalently, $\nu_{\mathrm{out}} \to -\nu_{\mathrm{out}}$. Since both the ν and $\widetilde{\nu}$ deformations are non-normalizable as $\phi \to +\infty$ they label backgrounds which are determined by the behavior at infinity and they do not fluctuate.

The coupling to T in (2.4) has important consequences. If $T(\phi)$ is nonzero the solution (2.6) becomes

$$\frac{C}{2\sqrt{2}} = \widetilde{\nu}t + \nu\int e^{-2T(\phi)}d\phi \tag{2.7}$$

(note, as a check that as $\phi \to +\infty$ it goes over to (2.6)). Consider the effect of $T(\phi) = \mu e^{-\phi}$ with positive μ on (2.7). At the strong coupling end $\phi \to -\infty$ the second term $\nu\int e^{-2T(\phi)}d\phi$ rapidly goes to zero, and the corresponding mode is normalizable. (Of course, it is not normalizable as $\phi \to +\infty$.) Hence it is a standard background deformation.

This is to be contrasted with the first term $\widetilde{\nu}t$. The norm of the small fluctuations which is derived from (2.4) is $\int d\phi e^{2T(\phi)}\delta C^2$, and hence it is not normalizable at $\phi \to -\infty$. Such a deformation, which is singular in the interior of the target space, can be present only if an object is present at its singularity. In our case, the relevant object is a D-brane which carries RR-charge. It sources the RR-flux $\widetilde{\nu}$.

For negative μ the situation is reversed. The RR-flux $\widetilde{\nu}$ is normalizable at $\phi \to -\infty$, and it does not need a D-brane source. However, the other flux $\nu\int e^{-2T(\phi)}d\phi$ needs D-branes at $-\infty$. This exchange in the behavior of the two fluxes under the change of the sign of μ is consistent with the S-duality symmetry.

As we vary μ from positive to negative values the physics has to change in a continuous fashion. This is particularly obvious in the asymptotic weak coupling end where the effects of nonzero μ are negligible. Therefore, we see here a phenomenon which has already been observed elsewhere [26 – 29]-[30, 19, 20]-[21], that RR-flux without D-branes can be continuously transformed to RR-flux carried by D-branes.

We should clarify the nature of these D-branes at infinity. In the worldsheet description these are the so called ZZ-branes [31]. Since they couple to the scalar C, the relevant branes are D-instantons. This means that our background flux represents a transition which is mediated by instantons. For positive μ we have $\widetilde{\nu}$ D-instantons per unit time and for negative μ we need ν such instantons per unit time. Although the number of such D-instantons is quantized, the numbers per unit time, ν or $\widetilde{\nu}$ do not have to be quantized.

Let us examine a background with generic ν and $\widetilde{\nu}$. It is easy to calculate the energy momentum tensor of that background as $\phi \to +\infty$. It is

$$T_{++} = \frac{1}{4\pi}\nu_{\rm in}^2 + \cdots , \qquad T_{--} = \frac{1}{4\pi}\nu_{\rm out}^2 + \cdots , \qquad T_{+-} = 0 + \cdots \tag{2.8}$$

Here the ellipses represent ϕ dependent corrections which are negligible as $\phi \to +\infty$. As far as the asymptotic lagrangian (2.4) is concerned, there is no problem with such a background. However, a crucial part of the story is that the dynamics is such that pulses get reflected from the $\phi = -\infty$ region. Furthermore the reflection from this region conserves energy. On the other hand, the incoming energy flux from $\mathcal{J}^-$ which is $\int dx^- T_{--}$ is not the same as the outgoing energy flux through $\mathcal{J}^+$ which is $\int dx^+ T_{++}$. (These integrals are infinite since we have a constant flux). Therefore, conservation of energy implies that we should either send in extra excitations from the past, or produce extra excitations in the future. For simplicity we can focus on the lowest energy excitation by adding excitations on the side that has the lower flux, so as to match the side with higher flux. The lowest energy configuration has

$$T_{++} = T_{--} = \frac{1}{4\pi}\max(\nu_{\rm in}^2, \nu_{\rm out}^2) + \cdots = \frac{1}{4\pi}(|\nu| + |\widetilde{\nu}|)^2 + \cdots . \tag{2.9}$$

Depending on whether $\nu_{\rm out}^2$ is bigger or smaller than $\nu_{\rm out}^2$, this is achieved by adding waves with $T_{t\phi} = \frac{1}{\pi}\nu\widetilde{\nu} + \cdots$ either in the past or in the future .

We are going to be interested in computing the scattering amplitude between a state in the past which is characterized by $\nu_{\rm in}$ and a state in the future which is characterized by $\nu_{\rm out}$. A full characterization of the states involves specifying the state for all the oscillators of the fields T and C. All we are doing in this section is to analyze the asymptotic region in order to understand which states we can send in and which states we expect to come out. Below, we will extend this discussion in the asymptotic region to the full system and will derive the exact expression for for the scattering amplitude. We will see that it depends only on $|\nu| + |\widetilde{\nu}|$.

Finally, we would like to comment on the 0B theory on a euclidean circle of radius R. The analytic continuation to euclidean space leads to real $\nu_E = i\nu$ and $\widetilde{\nu}_E = i\widetilde{\nu}$, where the subscript E denotes that these are the euclidean values. Here, in this euclidean time setup for positive μ the parameter ν_E is proportional to the number of instantons per unit euclidean time and therefore the parameter $q = 2\nu_E R$ is quantized (the precise normalization will be derived below). Similarly, for negative μ the parameter $\widetilde{q} = 2\widetilde{\nu}_E R$ is quantized. By continuity these two parameters are quantized for all μ.

2.2 0A

The discussion of the 0A string theory parallels that of the 0B theory. In fact, when the 0B theory is analytically continued to euclidean time and that coordinate is compactified, it is T-dual to the 0A theory.

The spectrum of the 0A theory consists of an NS-NS "tachyon" T, but the RR-scalar C is absent. It is replaced by two gauge fields $F_{t\phi}$ and $\widetilde{F}_{t\phi}$. These gauge fields have no propagating degrees of freedom, and only their zero momentum values can change.

Again, the theory has two discrete $\mathbf{Z}_2$ symmetries:

1. The charge conjugation symmetry which acts on the worldsheet theory as $(-1)^{F_L}$ acts on these fields as

$$\begin{aligned} T &\to T \\ F_{t\phi} &\to -F_{t\phi} \\ \widetilde{F}_{t\phi} &\to -\widetilde{F}_{t\phi}\,. \end{aligned} \tag{2.10}$$

2. The S-duality symmetry which acts on the worldsheet theory as $(-1)^{f_L}$ acts on the fields as

$$\begin{aligned} T &\to -T \\ F_{t\phi} &\to \widetilde{F}_{t\phi} \\ \widetilde{F}_{t\phi} &\to F_{t\phi}\,. \end{aligned} \tag{2.11}$$

The lagrangian (2.4) is replaced by

$$\mathcal{L} = \pi\alpha' \left(e^{2T} F_{t\phi}^2 + e^{-2T} \widetilde{F}_{t\phi}^2 \right) \tag{2.12}$$

which is invariant under the two symmetries (2.10)(2.11). The asymptotic solution of the equations of motion is $2\pi\alpha' F_{t\phi} = q$, $2\pi\alpha' \widetilde{F}_{t\phi} = \widetilde{q}$. Including the T dependent prefactors in (2.12) the solutions are

$$\begin{aligned} F_{t\phi} &= q e^{-2T} \\ \widetilde{F}_{t\phi} &= \widetilde{q} e^{2T}\,. \end{aligned} \tag{2.13}$$

For negative μ background $F_{t\phi}$ is singular at $\phi \to -\infty$ and is generated by D-branes at $\phi \to -\infty$, while the background $\widetilde{F}_{t\phi}$ is regular and does not need such branes. For positive μ the situation is reversed. These D-branes at infinity carry RR-electric charge. As in the 0B theory, these are charged ZZ-branes. However, unlike the D-instantons of the 0B theory the relevant branes couple to gauge field one forms, and therefore they are D0-branes. Hence, q and $\widetilde{q}$ are quantized.

Consider a background with generic quantized values of q and $\widetilde{q}$. By analogy with similar situations in critical string theory [32–34]-[35] we expect that such a background is possible only if we add to the system $q\widetilde{q}$ fundamental strings. This expectation can be

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derived by examining the coupling to the two form field B. Such a field does not have interesting dynamics in two dimensions, but its equation of motion shows that such strings must be present. This conclusion can also be derived by starting with the euclidean 0B theory on a circle. In the 0B theory we had to add energy flux to compensate the imbalance $T_{t\phi} \sim \frac{1}{\pi}\nu\widetilde{\nu}$. This translates, after rotation to euclidean space and T-duality, to adding $q\widetilde{q}$ strings in the 0A theory. Note that the sign of $q\widetilde{q}$ is correlated with the orientation of these strings in the two dimensional target space.

3. 0A Matrix model

In this section we consider the matrix model of the two dimensional 0A string theory [4] . This is a gauged matrix model which contains a complex matrix, m, which transforms in the bifundamental of $\mathrm{U}(N)_A \times \mathrm{U}(N)_B$. There are two ways of introducing fluxes. First, we can modify the gauge groups so that we start with $\mathrm{U}(N)_A \times \mathrm{U}(M)_B$ with $q = M - N$ (for simplicity assume that $q > 0$). This leads to $\widetilde{q} = 0$, $q \neq 0$. This corresponds to placing M charged ZZ branes and N anti-ZZ-branes at $\phi = -\infty$ and then letting the open string tachyon condense. It is clear from this description that as long as μ is below the barrier (in our conventions, $\mu < 0$), we will have q charged ZZ branes left over. So in this set up we end up describing the configuration with the flux that is sourced by D-branes. We expect that these ZZ branes will be stuck at the strong coupling end since a charged ZZ brane does not have an open string tachyon. One could consider charged D0 branes that move in the bulk of the two dimensional space [36] . We expect that these D0 branes will exist only in the non-singlet sector of the matrix model, since the euclidean boundary states that represent them contain a non-normalizable open string winding mode.

The second way to introduce flux is for $\widetilde{q} \neq 0$, $q = 0$. In this case we set $N = M$ and add to the matrix model a term of the form

$$S = S_0 + i\widetilde{q} \int (\mathrm{Tr}\, A - \mathrm{Tr}\, B) \tag{3.1}$$

where A and B are the gauge fields for the $\mathrm{U}(N)_A$ and $\mathrm{U}(N)_B$ gauge groups respectively. As shown in [4] this leads to a problem where the eigenvalues move in a complex plane, all with angular momentum $\widetilde{q}$. This can also be viewed as a special case of the general problem of coupling the matrix model to non-singlet representations. In this case we simply have a singlet representation of $\mathrm{SU}(N)_A \times \mathrm{SU}(N)_B$ which carries charge under the relative $\mathrm{U}(1)$ (which is generated by the difference between the generators of $\mathrm{U}(1)_A \subset \mathrm{U}(N)_A$ and of $\mathrm{U}(1)_B \subset \mathrm{U}(N)_B$). Below the barrier, $\mu < 0$, we can understand the origin of (3.1) as follows. As we explained above, the charged ZZ branes source the flux F. In this case the second flux $\widetilde{F}$ can be excited and leads to a smooth geometry. If we add ZZ branes the flux $\widetilde{F}$ leads to a Chern-Simons term on the worldvolume of the ZZ branes that produces (3.1) . This is the same type of coupling that leads to the usual Chern-Simons terms on D-brane worldvolumes in the ten dimensional superstring.

Surprisingly, once we reduce the problem to eigenvalues, the dynamics of these two cases is exactly the same [4]. Below we will slightly qualify this general comment.

We can now study the case with non-zero q and $\widetilde{q}$. The first naive idea is to consider again a rectangular matrix with $M = N + q$ and add the Chern-Simons term (3.1) . Let us assume for simplicity that $q > 0$. However, as we now explain, in this case the path integral vanishes. The matrix model degree of freedom, the matrix m, is not charged under the the diagonal U(1) generated by the sum of the generators of $\mathrm{U}(1)_A$ and $\mathrm{U}(1)_B$. Our normalization for these U(1)s is such that the fundamental representation of $\mathrm{SU}(N) \subset \mathrm{U}(N)$ has charge one (modulo N). On the other hand, the coupling (3.1) leads to charge $-q\widetilde{q}$ under diagonal U(1). Since this charge cannot be cancelled, the path integral vanishes. In other words we cannot obey the Gauss law for this gauge field.

In order to learn how to deal with this, let us return to the spacetime picture and understand what happens in string theory when we start with flux $\widetilde{q}$ and we attempt to put a charged ZZ brane. There is a coupling $S = -i\widetilde{q}\int B$ on the worldvolume of the ZZ brane, where B is the worldvolume U(1) gauge field. In order to cancel this charge we need to have $\widetilde{q}$ strings ending on the ZZ brane. If we have q charged ZZ branes we need to add $q\widetilde{q}$ strings ending on them. A similar situation has been encountered in various situations in [32 − 34]-[35]. So the matrix model that contains both fluxes necessarily involves the presence of a non-trivial representation of $\mathrm{U}(M)_B$ with M-ality $q\widetilde{q}$.

In summary, the matrix model with both fluxes is a $\mathrm{U}(N)_A \times \mathrm{U}(N+q)_B$ gauged matrix model

$$\int \mathcal{D}(A, B, m) e^{i\int dt\, \mathrm{Tr}[(D_0 m)^\dagger D_0 m + \frac{1}{2\alpha'} m^\dagger m]} e^{i\widetilde{q}\int(\mathrm{Tr}\, A - \mathrm{Tr}\, B)}\, \mathrm{Tr}_{\mathcal{R}}\, P e^{i\int B} \tag{3.2}$$

where $\mathcal{R}$ is a representation of $\mathrm{U}(N + q)_B$ with $q\widetilde{q}$ M-ality.[1]. We can now analyze this problem using the general procedure described in [37] We diagonalize the matrix m and we integrate out the gauge fields. Then we get an effective hamiltonian of the form[2]

$$\begin{aligned} H = \Bigg[\sum_{i=1}^{N} -\frac{1}{2}\frac{\partial^2}{\partial\rho_i} - \frac{1}{2}\rho_i^2 + \frac{1}{2}\frac{\widetilde{q}^2 + q^2 - \frac{1}{4}}{\rho_i^2} + 2\sum_{i<j\le N} \frac{\Pi_j{}^i \Pi_i{}^j}{(\rho_i^2 - \rho_j^2)} + \\ + \sum_{i=1}^{N} \frac{1}{\rho_i^2} \sum_{j>N} (\Pi_j{}^i \Pi_i{}^j + \Pi_i{}^j \Pi_j{}^i) \Bigg] P_0 \end{aligned} \tag{3.3}$$

where Π_i^j are the $\mathrm{U}(N+q)$ generators in the representation $\mathcal{R}$ and P_0 is a projector on the states obeying

$$\begin{aligned} \Pi_i^i &= 0 \qquad (\text{no sum}), \qquad i = 1, \cdots, N \\ \Pi_l{}^k &= \widetilde{q}\delta_l^k, \qquad\qquad\qquad l, k > N. \end{aligned} \tag{3.4}$$

The last condition implies that under the decomposition $\mathrm{U}(N + q) \to \mathrm{U}(N) \times \mathrm{U}(q)$ we select states that are singlets of $\mathrm{SU}(q)$ and carry $\mathrm{U}(1)_q \subset \mathrm{U}(q)$ charge $q\widetilde{q}$. The simplest way in which we can achieve this is by starting out with an $\mathrm{SU}(N+q)$ representation whose

[1] In principle we can also introduce a representation of $\mathrm{U}(N)_A$. In this case the constraint is that the M-ality of the representation of $\mathrm{U}(M)_B$ minus the N-ality of the representation of $\mathrm{U}(N)_A$ should be $q\widetilde{q}$.

[2] The apparent differences between this expression and the one in [37] are due to the fact that in [37] $\widetilde{q}$ is included as part of the U(1) charge of the representation.

Young tableaux contains q rows of length $\widetilde{q}$ (we assume that $\widetilde{q} > 0$), see figure 1 (a). Let us call this representation $\mathcal{R}_0$. In this case the state that is a singlet under SU(q) is also a singlet under SU(N) and obeys the two conditions (3.4) .

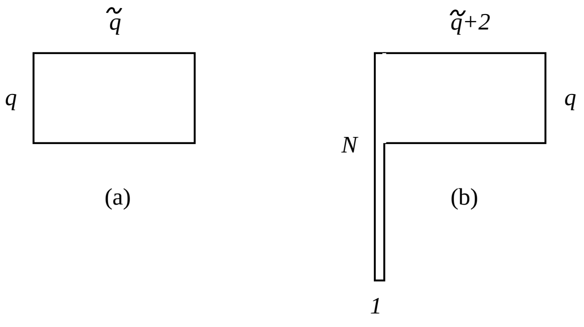

Figure 1: The Young tableaux (a) corresponds to the simplest representation $\mathcal{R}_0$ which leads to a nontrivial answer. The Young tableaux (b) is a more complicated representation which also contributes. The letters and numbers along the sides of the diagrams denote the number of boxes in that side.

We now need to consider the operator that appears in the hamiltonian (3.3)

$$Q_i = \sum_{j>N} (\Pi_j{}^i \Pi_i{}^j + \Pi_i{}^j \Pi_j{}^i) , \qquad i \leq N . \tag{3.5}$$

This operator transforms in the singlet of SU(q) and it decomposes as the singlet plus adjoint in SU(N). This implies that when we act on the single state that is SU(q) invariant in the representation $\mathcal{R}_0$ it can give us a state in the adjoint or the singlet of SU(N). Since the only state in $\mathcal{R}_0$ that is in the singlet of SU(q) is also in the singlet of SU(N), we conclude that the action of Q_i can only give us a singlet. So this action will just be a c-number. We can simply compute this c-number to be

$$Q_i = q\widetilde{q} \, . \tag{3.6}$$

Going back to the hamiltonian we find that it reduces to

$$H = \sum_{i=1}^{N} -\frac{1}{2}\frac{\partial^2}{\partial \rho_i} - \frac{1}{2}\rho_i^2 + \frac{1}{2}\frac{(\widetilde{q}+q)^2 - \frac{1}{4}}{\rho_i^2} \, . \tag{3.7}$$

Here we have assumed that q and $\widetilde{q}$ are positive. The same analysis can be repeated for the general case and we find that the dynamics depends only on

$$\widehat{q} = |q| + |\widetilde{q}| \, . \tag{3.8}$$

This is a surprising result from the point of view of the target space theory, as well as the matrix model.

Before we continue, let us explain what happens if our representation is a more general representation that contains a state obeying (3.4) . An example of a more general representation can be found in figure 1(b). In this case a state that obeys the second condition in (3.4) can be in the singlet or the adjoint of SU(N). The operator in (3.5) mixes the singlet with the adjoint of SU(N). So we expect that all these states will have properties that are similar to those encountered in general non-singlet representations as discussed in [37]. Such states have a divergent energy gap compared to the state that comes from the representation $\mathcal{R}_0$ in figure 1 (a). In the spacetime theory these states can be understood as states that, besides the $q\widetilde{q}$ strings ending on the charged ZZ brane, contain more string anti-string pairs. The fact that these strings stretch all the way to infinity is related to this divergence in the energy [37] . This divergence implies that these other states are in a different superselection sector. We would have found similar divergencies, due to extra strings, if we had also introduced a representation of the first group $\mathrm{U}(N)_A$. So from now on we will assume that we are adding simply the representation $\mathcal{R}_0$ in figure 1 (a).

Finally, let us present an alternate physical interpretation of the need for the representation $\mathcal{R}_0$. When the two kinds of branes/fluxes are present the system has massive open string fermions.[3] These can be added to the matrix model. Their quantization leads to several representations which ultimately, after the use of the Gauss law constraints, lead to $\mathcal{R}_0$.

The finite temperature partition function can then be computed as in [4] where the case $\widetilde{q}=0$ and arbitrary q was studied. We repeat this computation in appendix A. Our arguments that it is a function of $\widehat{q}=|q|+|\widetilde{q}|$ alow us to extend it to

$$\partial_\mu^3 \log \mathcal{Z}_{0A} = -\,\mathrm{Re}\,\frac{1}{2}\int_0^\infty \frac{dt}{t}\partial_\mu^3 e^{-(\widehat{q}+i\sqrt{2\alpha'}\mu)\frac{t}{2}} \frac{1}{\sinh\frac{t}{2}\sinh\sqrt{\frac{\alpha'}{2}}\frac{t}{2R}}\,. \tag{3.9}$$

Note that the integral in the right hand side converges. When we integrate this expression three times with respect to μ we need three integration constants — a q dependent second order polynomial in μ. We claim that the answer is

$$\begin{aligned}
\log \mathcal{Z}_{0A}(\mu,q,\widetilde{q},R) &= \Omega\,(y,r)+\Omega\,(\overline{y},r)+(2\pi R)\frac{\mu}{4}(|q|-|\widetilde{q}|)\\
y &= |q|+|\widetilde{q}|+i\sqrt{2\alpha'}\mu\\
r &= R\sqrt{\frac{2}{\alpha'}}
\end{aligned} \tag{3.10}$$

where the function $\Omega(y,r)$ is given by

$$\Omega(y,r)\equiv -\int_0^\infty \frac{dt}{t}\left[e^{-\frac{yt}{2}}\frac{1}{4\sinh\frac{t}{2}\sinh\frac{t}{2r}}-\frac{r}{t^2}+\frac{ry}{2t}+[\frac{1}{24}(r+\frac{1}{r})-\frac{ry^2}{8}]e^{-t}\right]. \tag{3.11}$$

We are interested in $\mathrm{Re}(y)=\widehat{q}\geq 0$ where this integral converges. (More generally, it converges for $\mathrm{Re}(y) > -(1+\frac{1}{r})$.) Therefore, Ω is an analytic function of y which is real along the positive real y axis. It is interesting that a closely related function appears in a totally different context in [38].[4]

The expression (3.10) for the partition function is one of the main results of this paper. Let us discuss it in more detail.

The last term in (3.10) depends on q and $\widetilde{q}$ separately and not only on $\widehat{q}=|q|+|\widetilde{q}|$. We will return to it below.

[3]These are similar to the fermions in the D0-D8 system, except that here they come from open strings stretched all the way to infinity, and hence they are infinitely massive.

[4]In terms of the function $G(x)$ in [38] our function is $\Omega(y,r)=C(b)+\log G(\frac{y}{2b}+\frac{Q}{2})$ where $b=\sqrt{r}$, $Q=b+b^{-1}$ and $C(b)$ is a constant independent of y.

It is surprising that up to this last term in (3.10) the complicated function $\log \mathcal{Z}_{0A}$ is given as a sum of a holomorphic and an anti-holomorphic functions. Correspondingly, the partition function $\mathcal{Z}_{0A}$ satisfies *holomorphic factorization*. The polynomial in μ which is not determined by (3.9) was fixed such that this holomorphic factorization is satisfied. In the next section we will present another computation of this partition function where some of this polynomial dependence on μ is independently determined.

It is straightforward to work out the asymptotic expansion of $\Omega(y,r)$ at large y with $\mathrm{Re}(y) \geq 0$

$$\Omega(y,r) = \left(\log \frac{y}{2} - \frac{3}{2}\right)\frac{ry^2}{8} - \frac{1}{24}\left(r + \frac{1}{r}\right)\log\frac{y}{2} - \frac{7r^2 + 10 + \frac{7}{r^2}}{1440}\frac{1}{ry^2} + \mathcal{O}\left(\frac{1}{y^4}\right) \qquad (3.12)$$

which leads to the following expression at large μ

$$\begin{aligned}\Omega(y = \widehat{q} + i\sqrt{2\alpha'}\mu, r) + \Omega(\overline{y} = \widehat{q} - i\sqrt{2\alpha'}\mu, r) = &\left(\frac{3}{2} - \log\left(\sqrt{\frac{\alpha'}{2}}|\mu|\right)\right)\frac{\alpha' r\mu^2}{2} - \frac{2\pi R|\mu|\widehat{q}}{4} + \\ &+\left(\frac{\widehat{q}^2 r}{4} - \frac{1}{12}(r + \frac{1}{r})\right)\log\left(\sqrt{\frac{\alpha'}{2}}|\mu|\right) + \\ &+\frac{\frac{7}{r^2} + 10 + 7r^2 + 15\widehat{q}^2 r(\widehat{q}^2 r - 2\left(r + \frac{1}{r}\right))}{1440\alpha' r\mu^2} + \\ &+\mathcal{O}\left(\frac{1}{\mu^4}\right). \end{aligned} \qquad (3.13)$$

In the worldsheet genus expansion these terms have the following interpretation. The first term corresponds to the sphere contribution. The scaling of the second term which is proportional to $\widehat{q}$ shows that it arises from a disk diagram. We will return to this term below. The $\widehat{q}$ independent term in the second line is the contribution of the torus and the term proportional to $\widehat{q}^2$ corresponds to a sphere with two RR insertions, or an annulus. Higher orders can be discussed similarly.

Let us focus on the term $-2\pi R|\mu|\widehat{q}/4$ in (3.13), which comes from the first term in (3.12) . Despite appearance, because of the absolute value sign on μ, it should not be discarded as an un-interesting analytic term. We can now understand the role of the last term in (3.10). Combining it with $-2\pi R|\mu|\widehat{q}/4$ we have

$$-\frac{2\pi R|\mu|}{4}(|q| + |\widetilde{q}|) + \frac{2\pi R\mu}{4}(|q| - |\widetilde{q}|) = \begin{cases} -\pi R\mu|\widetilde{q}| & \mu > 0 \\ -\pi R|\mu||q| & \mu < 0 \end{cases} \qquad (3.14)$$

We interpret this as the contribution of the disk amplitude of the charged ZZ-branes. For positive μ we have $|\widetilde{q}|$ ZZ-branes and for negative μ we have $|q|$ ZZ-branes. Each has energy $|\mu|/2$. (Recall that the energy of a brane-anti-brane pair is equal to $|\mu|$. So the energy of a single charged D-brane should be equal to $|\mu|/2$.). From the point of view of the 0A matrix model we needed to introduce the last term in (3.10) "by hand" in order to obey (3.14) . This is an analytic term in μ, so it is, in principle, possible to introduce it. However, we will see that this term emerges naturally from the 0B matrix model.

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So after including the analytic term we find precisely the expected behavior for the free energy. For $\mu < 0$ we have D-branes that produce flux proportional to q and there are no terms that have odd powers in $\widetilde{q}$ in the asymptotic expansion. On the other hand for $\mu > 0$ we have the opposite situation, since now the flux $\widetilde{q}$ is sourced by D-branes.

Despite this simple physical interpretation, our result is still surprising. With the exception of the disk term (3.14) the semiclassical expansion includes only even powers of μ and q and $\widetilde{q}$. This means that there are no contributions from worldsheets with odd number of boundaries. For positive μ this is the expected result when $\widetilde{q} = 0$ and there are no ZZ-branes. Similarly, for negative μ this is the expected result when $q = 0$. The dependence on q and $\widetilde{q}$ through $\widehat{q} = |q| + |\widetilde{q}|$ together with these expected results guarantee that, with the exception of the disk (3.14), there are no contributions from surfaces with odd number of boundaries. We do not have a worldsheet or spacetime interpretation of this surprising result.

4. 0B Matrix model

4.1 lorentzian 0B model

In this section we consider the 0B matrix model which consists of a hermitian matrix model with an inverted harmonic oscillator potential such that in the free fermion description we fill the two sides of the inverted harmonic oscillator potential [3, 4].

To analyze this problem it is useful to realize that the asymptotic region of the weak coupling end in the target space geometry is associated to the asymptotic region of the Fermi sea far away from the maximum of the potential. So the two RR fluxes ν, $\widetilde{\nu}$ that we discussed in section 2 are associated with the Fermi levels of the fermions on the two sides of the potential. Far from the maximum of the potential we can approximate the fermions as relativistic fermions since the depth of the Fermi sea is much larger than any finite energy we consider. It is also possible, and useful, to consider a basis for the inverted harmonic oscillator problem where the fermions are exactly relativistic [25]. We review and extend this formalism in detail in appendix A. There are actually two possible bases, which are naturally associated to the coordinates $u = \frac{1}{\sqrt{2}}(p - x)$ and $s = \frac{1}{\sqrt{2}}(p + x)$. These are the bases of *in* and *out* states and the S-matrix gives the relation between them. This relation is simply a Fourier transform.

So when we think about the asymptotic states we should think in terms of relativistic fermions. The asymptotic states live in the *in* and *out* Hilbert spaces of the fermions that are going towards the maximum of the potential or away from it. Each of these Hilbert spaces is described by two complex fermions

$$\psi^{\rm in}_{\pm}\,, \qquad \psi^{{\rm in}\,\dagger\pm}\,, \qquad \psi^{\rm out}_{\pm}\,, \qquad \psi^{{\rm out}\,\dagger\pm}\,. \tag{4.1}$$

The $+/-$ indices denote fermions that are moving towards the right/left. It is very important not to confuse right and left moving matrix model fermions (denoted here by $+/-$), which are moving to the right or left in eigenvalue space, with left and right movers in

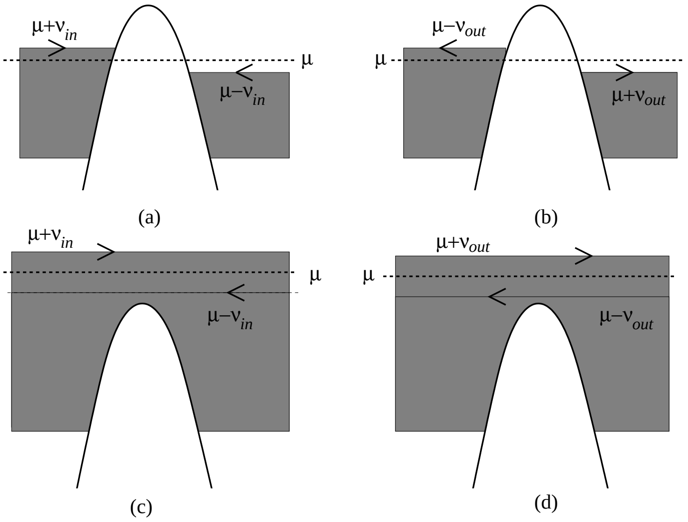

Figure 2: Configurations with generic $\nu_{in,out}$ describe scattering amplitudes. Figures (a) and (b) describe scattering below the potential barrier $\mu < 0$, and figures (c) and (d) describe scattering above the potential barrier $\mu > 0$. Figures (a) and (c) describe the initial configurations, while (b) and (d) describe the final configuration. The dotted line represents the Fermi level characterized by μ. (Even though we have represented the Fermi surface reaching all the way to the potential wall, we really should think of these configurations as asymptotic states, or as states in the *in* or *out* basis defined in the text.) Note that, as in the figure, the dominant scattering amplitudes for $\mu > 0$ have $\nu_{\rm in} \approx \nu_{\rm out}$; i.e. $\widetilde{\nu} \approx 0$, while for $\mu < 0$ they have $\nu_{\rm in} \approx -\nu_{\rm out}$; i.e. $\nu \approx 0$. Note that in (b) $\nu_{\rm out} < 0$.

spacetime, which are related to incoming or outgoing states.[5] Our notation emphasizes the charge of the fermion under the U(1) current which measures the number of right minus left moving fermions. This is the current associated to the scalar C in spacetime. More precisely,

$$\begin{aligned} i(\partial_t + \partial_\phi)C &\sim \psi^{\rm in\dagger +}\psi_+^{\rm in} - \psi^{\rm in\dagger -}\psi_-^{\rm in} \\ i(\partial_t - \partial_\phi)C &\sim \psi^{\rm out\dagger +}\psi_+^{\rm out} - \psi^{\rm out\dagger -}\psi_-^{\rm out}\,. \end{aligned} \tag{4.2}$$

In principle we can specify freely the four Fermi levels of these four fermions. The fact that fermion number is conserved implies one relation between these four levels. So we have three independent levels which denote by μ, $\nu_{\rm in}$ and $\nu_{\rm out}$. These are defined by saying that $\mu \pm \nu_{in,out}$ are the Fermi levels associated to the right and left moving incoming and outgoing fermions (see figure 2).[6]

In the case that we set $|\nu_{\rm in}| \neq |\nu_{\rm out}|$ we find that the incoming energy flux is not the same as the outgoing flux. Therefore we will need to add additional excitations. This is the same as in the discussion of the spacetime theory in section 2.

[5] Also, in the matrix model, do not confuse right and left *moving* fermions with fermions that are to the left and right side of the potential. For example, the *in* right moving fermion is to the left of the potential.

[6] When the time is rotated to euclidean space we must also rotate $\nu \to i\nu$. This leads to an imaginary shift of the Fermi surface. This is consistent with the analysis of the $\widehat{c} < 1$ systems which are similar to euclidean $\widehat{c} = 1$ where the RR flux was interpreted as an imaginary shift of the Fermi surface [19, 20]-[21].

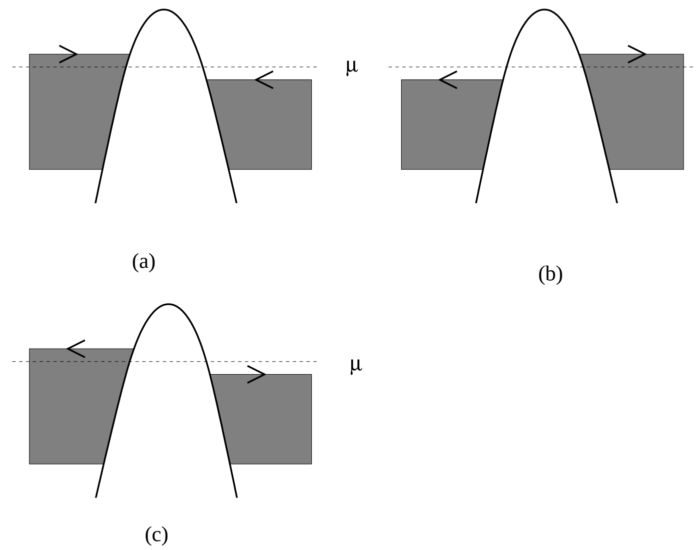

Figure 3: In (a) we see an initial configuration of incoming fermions with $\nu_{\rm in} > 0$. In (b),(c) we see a outgoing configurations with $\nu_{\rm out} > 0$ and $\nu_{\rm out} < 0$ respectively. For the combination (a) (b) we have $\nu_{\rm out} = \nu_{\rm in}$ and therefore $\widetilde{\nu} = 0$, $\nu = \nu_{\rm in}$. On the other hand for the combination (a), (c) we have $\nu_{\rm out} = -\nu_{\rm in}$ or $\nu = 0$, $\widetilde{\nu} = \nu_{\rm in}$.

All the information about the inverted harmonic oscillator potential is contained in the map between in and out states (see appendix A)

$$\psi^{\rm out}_{a,\epsilon} = \sum_{b=\pm 1} \mathcal{S}_a^b(\epsilon)\psi^{\rm in}_{b,\epsilon} = \frac{\Gamma(\frac{1}{2} - i\sqrt{2\alpha'}\epsilon)}{\sqrt{2\pi}} \sum_{b=\pm 1} e^{i\frac{\pi}{2}ab(\frac{1}{2}-i\sqrt{2\alpha'}\epsilon)}\psi^{\rm in}_{b,\epsilon} \tag{4.3}$$

where $a, b = \pm 1$ and $\psi^{in,out}_{a,\epsilon}$ denote the annihilation operator for a fermion of energy ϵ.

We will now compute the transition amplitude between an *in* state with Fermi levels μ, $\nu_{\rm in}$ to an *out* state with Fermi levels μ, $\nu_{\rm out}$

$$\mathcal{A} = \langle out(\mu, \nu_{\rm out})|in(\mu, \nu_{\rm in})\rangle \tag{4.4}$$

and interpret it as the partition function of the 0B theory with nonzero ν and $\widetilde{\nu}$: $\mathcal{A} = \mathcal{Z}_{0B}(\mu, \nu, \widetilde{\nu})$.

For simplicity let us first assume that $\nu_{\rm in} = \nu_{\rm out} = \nu > 0$. Consider the *in* state. The right moving fermions, which are created by $\psi^{\rm in+}$ are filled up to the Fermi level $\mu + \nu$, while the left moving fermions, created by $\psi^{\rm in-}$ are filled up to the Fermi level $\mu - \nu$. The same is true for the *out* fermions. So the overlap is given by[7]

$$\mathcal{A} = \prod_{-\infty<\epsilon_n<\mu-\nu} [\mathcal{S}^+_+(\epsilon_n)\mathcal{S}^-_-(\epsilon_n) - \mathcal{S}^+_-(\epsilon_n)\mathcal{S}^-_+(\epsilon_n)] \prod_{\mu-\nu<\epsilon_m<\mu+\nu} \mathcal{S}^+_+(\epsilon_m) \tag{4.5}$$

where we have regularized the continuum by putting the system on a circle of length L so that the density of states is $dn = \frac{L}{2\pi}d\epsilon$. We have used that up to the energy $\mu - \nu_{\rm in}$

[7]This expression for $\mathcal{A}$ suffers from a phase ambiguity. We can transform the incoming and outgoing, left and right moving Hilbert spaces by arbitrary energy independent phases. These four phases can be used to remove terms linear in μ, ν (and later $\widetilde{\mu}$) in $\log\mathcal{A}$.

both states are occupied. Note that the Fermi statistics produces the determinant of $\mathcal{S}$ for these states. On the other hand, for energies in the band between $\mu - \nu_{\rm in}$ and $\mu + \nu_{\rm in}$ we have only the amplitude for an incoming right fermion going to an outgoing right fermion. Taking the logarithm of (4.5) and expressing the resulting sums in terms of integrals we obtain[8]

$$\begin{aligned}\log\mathcal{A} = &\frac{L}{2\pi}\Bigg[\int_{-\Lambda}^{\mu-\nu} d\epsilon \log\left(\Gamma\left(\frac{1}{2} - i\epsilon\right)\Big/\Gamma\left(\frac{1}{2}+i\epsilon\right)\right) + \int_{\mu-\nu}^{\mu+\nu} d\epsilon \log\left(\Gamma\left(\frac{1}{2} - i\epsilon\right)\Big/\sqrt{2\pi}\right) + \\ &\qquad + \frac{\pi}{2}\int_{\mu-\nu}^{\mu+\nu} d\epsilon\ \epsilon\Bigg] \\ = &\frac{L}{2\pi}\left[\int_{-\Lambda}^{\mu+\nu} d\epsilon \log\left(\frac{\Gamma(\frac{1}{2} - i\epsilon)}{\sqrt{2\pi}}\right) - \int_{-\Lambda}^{\mu-\nu} d\epsilon \log\left(\frac{\Gamma(\frac{1}{2} + i\epsilon)}{\sqrt{2\pi}}\right) + \pi\mu\nu\right]. \end{aligned} \tag{4.6}$$

Here Λ is a cutoff on the bottom of the Fermi sea. Using

$$\log\left(\Gamma\left(\frac{1}{2} + z\right)/\sqrt{2\pi}\right) = \int_0^\infty \frac{dt}{t}\left[\frac{e^{-zt}}{2\sinh\frac{t}{2}} - \frac{1}{t} + ze^{-t}\right] \tag{4.7}$$

and defining $\Xi(y)$ as related to the large radius limit of $\Omega(y,r)$ of (3.11)

$$\Xi(y) \equiv \lim_{r\to\infty}\frac{\Omega(y,r)}{2\pi r} = -\frac{1}{2\pi}\int_0^\infty \frac{dt}{t}\left[e^{\frac{-yt}{2}}\frac{1}{2t\sinh\frac{t}{2}} - \frac{1}{t^2} + \frac{y}{2t} + \left(\frac{1}{24} - \frac{y^2}{8}\right)e^{-t}\right] \tag{4.8}$$

(4.6) becomes

$$\log\mathcal{A} = -iL\left[-\Xi(-2i(\mu+|\nu|)) - \Xi(2i(\mu-|\nu|)) + i\frac{1}{2}\mu|\nu|\right] \tag{4.9}$$

where we suppressed a Λ dependent imaginary constant and extended the answer to negative ν using the charge conjugation symmetry (left-right symmetry in the matrix model) $\nu \to -\nu$.

Note that the 0B free energy at $\nu = 0$ in the non-compact limit is given by

$$F_{0B} = -\lim_{\beta\to\infty}\frac{\log\mathcal{Z}_{0B}}{\beta} = \lim_{T_L\to\infty}\frac{\log\mathcal{A}}{-iT_L} = -\Xi(-2i\mu) - \Xi(2i\mu)\,. \tag{4.10}$$

We have interpreted the cutoff L as the length of lorentzian time, $T_L = L$ since, for large L, this is the time it takes for the *in* state to come out from the scattering region. In other words the spatial cutoff L is a good approximation for times which are of order L.

From the asymptotic expansion (3.12) we can find the asymptotic expansion of Ξ.

$$\Xi(y) = \frac{1}{2\pi}\left[\left(\log\frac{y}{2} - \frac{3}{2}\right)\frac{y^2}{8} - \frac{1}{24}\log\frac{y}{2} - \frac{7}{1440y^2} + \mathcal{O}\left(\frac{1}{y^4}\right)\right]. \tag{4.11}$$

This implies that the only perturbative terms in the imaginary part arise from the logarithms in (4.11)

$$\mathrm{Im}[\Xi(-2i\mu)] \approx \frac{|\mu|\mu}{8}\left(1 + \frac{1}{12}\frac{1}{\mu^2}\right) \tag{4.12}$$

[8]We set $\alpha' = \frac{1}{2}$.

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(Because of the dependence on the absolute value of μ they are not analytic and should be kept.) Thus, going back to (4.9) , we find that the leading contribution to the real part of the log of the amplitude is

$$\begin{aligned}
\mathrm{Re}[\log \mathcal{A}] \approx T_L \Big[& -\frac{1}{8}|\mu+|\nu||(\mu+|\nu|)\left(1+\frac{1}{12}\frac{1}{(\mu+|\nu|)^2}\right)+ \\
& +\frac{1}{8}|\mu-|\nu||(\mu-|\nu|)\left(1+\frac{1}{12}\frac{1}{(\mu-|\nu|)^2}\right)+\frac{1}{2}\mu|\nu|\Big] \\
= & \begin{cases} 0 & \mu\pm|\nu|>0 \\ T_L\mu|\nu| & \mu\pm|\nu|<0 \end{cases}
\end{aligned} \tag{4.13}$$

So we see that for $\mu \pm |\nu| > 0$ the leading approximation to $\log \mathcal{A}$ is purely imaginary. Here we are in a configuration where the two Fermi seas are above the barrier (see figure 2 (c)/(d)), and the amplitudes is dominated by the leading order transmission over the barrier, as expected. On the other hand, for $\mu\pm|\nu| < 0$, there is a negative real contribution. This implies that this processes is suppressed. Here the Fermi seas are below the barrier, and the in/out states are as in figure 3 (a)/(b). (Figure 2 (a)/(b) depict a configuration with nonzero $\widetilde{\nu}$.) In order to obey these boundary conditions we need to have tunneling processes. We need of the order of $|\nu|$ tunneling events per unit time. Each tunneling event contributes a factor of $e^{\mu\pi}$ (recall, $\mu < 0$), which is the contribution of a charged D-instanton ZZ brane. The number of such factors depends on the total time T_L as $T_L|\nu|/\pi$. This leads to (4.13) . Note that the effects of the instantons do not exponentiate since we are looking at a very special process where only a definite number of instantons could contribute. Of course one could also study processes where $\mu - |\nu| < 0 < \mu + |\nu|$. Then, the second term in (4.12) also contributes.

The expression (4.9) for the partition function $\mathcal{A}$ has a few interesting consequences. The two terms $-\Xi(-2i(\mu + |\nu|)$ and $-\Xi(2i(\mu - |\nu|)$ can be interpreted as the contribution of the fermions in the left and the right side of the potential. Therefore, either $-\Xi(2i\mu)$ or $-\Xi(-2i\mu)$ can be viewed as a nonperturbative definition of the free energy of the bosonic $c = 1$ system with Fermi level μ. (Recall, the bosonic system has fermions only in one side of the potential.) From this perspective the problem with the $c = 1$ system is that $-\Xi(\pm 2i\mu)$ are complex. The sign ambiguity in the definition changes the sign of the imaginary part which signals the instability of the system.

The expression (4.9) also gives an intuitive explanation of the holomorphic factorization we have seen before. Up to the simple term which depends on $\mu|\nu|$ the partition function factorizes as a product of $\exp(iL\Xi(-2i(\mu+|\nu|)$ and $\exp(iL\Xi(2i(\mu-|\nu|)$ which are associated with the incoming fermions from the left and the right side of the potential. Our definition of the RR-flux is such that it does not mix these two kids of fermions, and therefore we can specify independent Fermi levels for them, $\mu \pm |\nu|$. In euclidean space $|\nu| \to i|\nu|$ and therefore this separation explains the holomorphic factorization we discussed above. We will soon add nonzero $\widetilde{\nu}$, and will study the problem with a euclidean time circle. The separation of these modes will persist. It underlies the holomorphic factorization of the partition function.

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Repeating this analysis for $\nu_{\rm in} = -\nu_{\rm out} = \widetilde{\nu}$ we find an answer that is very similar to (4.9) except that the last term changes sign

$$\log \mathcal{A} = -iL \left[-\Xi(-2i(\mu + |\widetilde{\nu}|)) - \Xi(2i(\mu - |\widetilde{\nu}|)) - i\frac{1}{2}\mu|\widetilde{\nu}| \right]. \tag{4.14}$$

In this case the real part is small for sufficiently large negative μ but it is behaves as $-L\mu|\widetilde{\nu}|$ for large positive μ. This is consistent with the duality symmetry $\mu \to -\mu$, $\nu \leftrightarrow \widetilde{\nu}$.

In the case that $\nu_{\rm in}^2 \neq \nu_{\rm out}^2$ we have to insert extra asymptotic states in order to balance the energy flux. We will do this in more detail in the euclidean computation in the next subsection.

4.2 Computation at finite R

In this section we consider the finite temperature partition function, where the time direction has period $\beta = 2\pi R$. Configurations with RR fluxes correspond to configurations where the field C or its dual have winding along the euclidean time direction. We have said that the field C is at the self dual radius, $C \sim C + 2\pi\sqrt{2}$ with the normalization (2.4) . This implies the following quantization condition for the constant part of the euclidean time derivative

$$\partial_\tau C = -i2\sqrt{2}\widetilde{\nu} = \sqrt{2}\frac{\widetilde{q}}{R}, \qquad \widetilde{q} \in \mathbf{Z}. \tag{4.15}$$

We can similarly think about the quantization condition for, $\widetilde{C}$, the dual of C. This gives

$$\partial_\phi C = 2\sqrt{2}\nu = i\sqrt{2}\frac{q}{R}, \qquad q \in \mathbf{Z}. \tag{4.16}$$

We then define

$$q_{\rm in} = q + \widetilde{q}, \qquad q_{\rm out} = q - \widetilde{q}. \tag{4.17}$$

Note that $q, \widetilde{q}, q_{\rm in}, q_{\rm out}$ are integer, but $q_{\rm in} - q_{\rm out}$ is always even. In these conventions an *in* right moving fermion has $q_{\rm in} = 1$. Note that we can consider *in* states where we have a left moving spin field and a right moving spin field. If the charge of the spin fields are opposite this configuration gives us $q_{\rm in} = 1$. But then we should also have spin fields in the *out* state since (4.17) (4.15) (4.16) imply that $q^{in,out}$ are either both odd or both even.

There are two closely related ways of thinking about the euclidean computation. One is to view it as an analytic continuation of the lorentzian scattering computation (4.4) . The only difference is that the asymptotic regions now look like a cylinder. So we think of the *in* and *out* states as living on a cylinder and we expand them in fourier modes along the compact direction. The analytic continuation of (A.25) gives the relation between *in* and *out* fields. The resulting relation can be summarized as (we continue to set $\alpha' = \frac{1}{2}$)

$$\begin{aligned} \langle \psi_r^{out\dagger\ a} \psi_{b,\ -s}^{\rm in} \rangle &= \delta_{r,s} \frac{\Gamma(\frac{1}{2} + i\mu + s/R)}{\sqrt{2\pi}} e^{\frac{\pi}{2}(\mu - i\frac{s}{R})ab - i\frac{\pi}{4}ab} \\ \langle \psi_{a,\ r}^{\rm out} \psi_{-s}^{in\dagger\ b} \rangle &= \delta_{r,s} \frac{\Gamma(\frac{1}{2} - i\mu + s/R)}{\sqrt{2\pi}} e^{\frac{\pi}{2}(\mu + i\frac{s}{R})ab + i\frac{\pi}{4}ab} \end{aligned} \tag{4.18}$$

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where $r, s \in \mathbf{Z} + \frac{1}{2}$, $r, s > 0$. When we do this analytic continuation of (A.25) we might be a bit unsure about the sign for s in the right hand side. This sign is determined by doing the analytic continuation of the fields carefully and demanding that the mode, $\psi^{\rm in}_{-s}$ does not annihilate the vacuum, $\psi_{-s}|0\rangle \neq 0$ for positive s (in our conventions $\psi_s^{in/out}|0\rangle = 0$ for $s > 0$). In this description we are interested in computing an inner product of the form

$$\langle \Psi_{\rm out}|\Psi_{\rm in}\rangle \,. \tag{4.19}$$

Notice that from the target space viewpoint, the states $\Psi_{\rm in}$ and $\Psi_{\rm out}$ are determined by choosing the non-normalizable behavior of the anti-holomporphic and the holomorphic parts of the target space fields T and C near the boundary. As usual, this correspondence involves bosonization of the fermions and an identification of the modes of the bosonic field with the modes of the T and C fields.

The other way of thinking about the problem consists in viewing the problem as defined on a half cylinder where the *in* and *out* fields are antiholomorphic and holomorphic fields respectively. We impose boundary condition at the asymptotic end of the cylinder by specifying a state in the Hilbert space for fermions on a cylinder. In the capped end of the semi infinite cylinder we insert a boundary state which encodes the effects of the scattering amplitude. This boundary state is also computed by analytically continuing (A.25) . It is clear that both pictures are equivalent and which one we choose is a matter of taste.

Let us consider a configuration with general $q_{\rm in}$ and $q_{\rm out}$, and first consider the case

$$q_{\rm in} \geq q_{\rm out} \geq 0 \,. \tag{4.20}$$

For simplicity, let us limit ourselves to $q_{\rm in}, q_{\rm out} \in 2\mathbf{Z}$.[9] So we will have a state in the *in* Hilbert space of dimension $\Delta_{\rm in}$. Since our problem is invariant under translations in euclidean time, we need that $\Delta_{\rm out} = \Delta_{\rm in}$. We will be interested in considering the state with lowest $\Delta_{\rm in}$ since this is the state that corresponds to exciting only the constant part of the RR field strength (the gradient of $(\partial_\phi - i\partial_t)C \sim q_{\rm in}$). This lowest dimension state is

$$\Delta_{\rm in} = \frac{q_{\rm in}^2}{4} = \Delta_{\rm out} \tag{4.21}$$

which corresponds to the state

$$|\Psi\rangle_{\rm in} = \prod_{l=1}^{q_{\rm in}/2} \psi^{{\rm in}\dagger +}_{-l+1/2}\psi^{\rm in}_{-,-l+\frac{1}{2}}|0\rangle \,. \tag{4.22}$$

States with higher dimension, with the same $q_{\rm in}$ correspond to exciting other oscillator modes of T and C.

In the out Hilbert space we need a state with the same conformal dimension. Since the S-matrix is given by the product of one body S-matrices it is clear that we need as many fermions and holes in the *out* Hilbert space as we have in the *in* Hilbert space. Since our

[9] This case is simpler because we do not need to introduce spin fields.

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system is time translation invariant, the amplitudes are diagonal in the mode number. So the non-zero amplitudes have the form

$$\langle 0| \prod_{l=1}^{q_{\rm in}/2} \psi^{out\dagger \ a_n}_{l-\frac{1}{2}} \psi^{\rm out}_{b_n, l-\frac{1}{2}} \psi^{in\dagger +}_{-l+\frac{1}{2}} \psi^{\rm in}_{-,-l+\frac{1}{2}} |0\rangle \,. \tag{4.23}$$

Since the charge of the out state has to be $q_{\rm out}$ we need that $\sum b_n - a_n = q_{\rm out}$.[10] There are several ways to assign values of a_n and b_n. Of the $q_{\rm in}$ out-fermions, $q = (q_{\rm in} + q_{\rm out})/2$ should have *out* charge minus one and $\widetilde{q} = (q_{\rm in} - q_{\rm out})/2$ should have *out* charge plus one. The number of possibilities of achieving this is

$$N(q, \widetilde{q}) = \frac{q_{\rm in}!}{[\frac{1}{2}(q_{\rm in} + q_{\rm out})]![\frac{1}{2}(q_{\rm in} - q_{\rm out})]!} = \frac{(q + \widetilde{q})!}{q!\widetilde{q}!} \,. \tag{4.24}$$

Using (4.18) we can compute (4.23) and obtain

$$\begin{aligned} \mathcal{A}(\mu, q_{\rm in}, q_{\rm out}) &= e^{i\varphi(a_n, b_n, R)} \prod_{n=1}^{q_{\rm in}/2} \frac{|\Gamma(\frac{1}{2} - i\mu + (n - \frac{1}{2})/R)|^2}{2\pi} e^{\frac{-\pi\mu(a_n - b_n)}{2}} \mathcal{A}(\mu, 0, 0) \\ &= e^{i\varphi(a_n, b_n, R)} e^{\frac{\pi\mu q_{\rm out}}{2}} \prod_{n=1}^{q_{\rm in}/2} \frac{|\Gamma(\frac{1}{2} - i\mu + (n - \frac{1}{2})/R)|^2}{2\pi} \mathcal{A}(\mu, 0, 0) \end{aligned} \tag{4.25}$$

where we have used $q_{\rm out} = \sum (b_n - a_n)$. Note that up to the phase $e^{i\varphi(a_n, b_n, R)}$ the answer does not depend on the particular operator among all the $N(q_{\rm in}, q_{\rm out})$ operators in (4.24) with the same charges.[11] It will be important below that this phase is independent of μ.

It is straightforward to extend the computation (4.25) to values of $q_{\rm in}$ and $q_{\rm out}$ which do not satisfy (4.20). The answer is expressed most easily in terms of q and $\widetilde{q}$

$$\mathcal{A}(\mu, q_{\rm in}, q_{\rm out}) = e^{i\varphi(a_n, b_n, R)} e^{\frac{\pi\mu}{2}(|q| - |\widetilde{q}|)} \prod_{n=1}^{\widehat{q}/2} \frac{|\Gamma(\frac{1}{2} + i\mu + \frac{n - \frac{1}{2}}{R})|^2}{2\pi} \mathcal{Z}_{0B}(\mu, q = \widetilde{q} = 0, R) \tag{4.26}$$

where $\widehat{q} = |q| + |\widetilde{q}| = q_{\rm in}$ for the case (4.20) .

Using (4.7) and the expression for $\Omega(y, r)$ (3.11) we find for even $\widehat{q} = 2k$.[12]

$$\begin{aligned} \Omega\left(y = \frac{2k}{R} + 2i\mu, R\right) &- \Omega(y = 0 + 2i\mu, R) = \\ &= -\int_0^\infty \frac{dt}{t} \left[e^{-i\mu t} \frac{e^{-\frac{kt}{R}} - 1}{4\sinh\frac{t}{2}\sinh\frac{t}{2R}} + \frac{k}{t} - \left(\frac{k^2}{2R} + ik\mu\right) e^{-t} \right] \\ &= \sum_{n=1}^{k} \int_0^\infty \frac{dt}{t} \left[\frac{e^{-(\frac{2n-1}{2R} + i\mu)t}}{2\sinh\frac{t}{2}} - \frac{1}{t} + \left(\frac{2n-1}{2R} + i\mu\right) e^{-t} \right] \end{aligned}$$

[10]Notice that we are defining the charge as the charge of the ket, which is minus the charge of the *out* operators explicitly appearing in (4.23).

[11]The technical reason for this is the fact that the difference between the right to right vs right to left amplitudes (A.25) is a simple exponential.

[12]This is the same as the recursion relations found for the function $G(x)$ in [38]

$$= \sum_{n=1}^{k} \log \left(\frac{\Gamma\left(\frac{1}{2} + i\mu + \frac{n-\frac{1}{2}}{R}\right)}{\sqrt{2\pi}} \right). \tag{4.27}$$

Using this relation and the expression for $\mathcal{Z}_{0B}(\mu, q = \widetilde{q} = 0)$ (see appendix A), (4.26) can be written as

$$\begin{aligned} \log \mathcal{Z}_{0B}(\mu, q, \widetilde{q}, R) &= \log \mathcal{A}(\mu, q_{\rm in}, q_{\rm out}) \\ &= i\varphi(a_n, b_n, R) + \frac{\pi\mu}{2}(|q| - |\widetilde{q}|)\Omega\left(y = \frac{\widehat{q}}{R} + 2i\mu, R\right) + \\ &\quad + \Omega\left(y = \frac{\widehat{q}}{R} - 2i\mu, R\right) \end{aligned} \tag{4.28}$$

which is our final expression for the free energy.

After we apply T-duality

$$R_B = \frac{\alpha'}{R_A}, \qquad \mu_B = \frac{R_A}{\sqrt{2\alpha'}}\mu_A \tag{4.29}$$

we find that (4.28) becomes the same as (3.10), up to the phase and analytic terms in μ proportional to $\log R$. These terms are related to the fact that we need to change the UV cutoff Λ when we perform T-duality (see the appendix).

Note that this 0B computation produces naturally the term that involves $(|q| - |\widetilde{q}|)$ while in the 0A problem we had to introduce this term "by hand" in order to match the expected asymptotic behavior.

Notice that this procedure produces answers which are consistent with T-duality, while previous studies did not.

The answer (4.26) has the expected symmetries: $A(\mu, q, \widetilde{q}) = A(\mu, -q, -\widetilde{q}) = A(-\mu, \widetilde{q}, q) = A(\mu, -q, \widetilde{q})^*$. They follow from the two $\mathbf{Z}_2$ symmetries and time reversal.

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A. Chiral quantization of matrix models

The purpose of this appendix is to derive some known results about the $\widehat{c} = 1$ 0A and 0B matrix models. We will use a formalism which was first introduced in [25] and was later used and elaborated on in [39–46]-[47]. It highlights the chiral nature of the problem and the scattering from and to null infinities. One of the advantages of this formalism is that the theory is expressed in terms of *free relativistic fermions.* The nontrivial scattering appears as a nonlocal transform between the incoming and the outgoing descriptions. The parabolic cylinder functions of the inverted harmonic oscillator are replaced by simple wave functions in the $p \pm x$ representation. $p \pm x$ are the analog of creation and annihilation

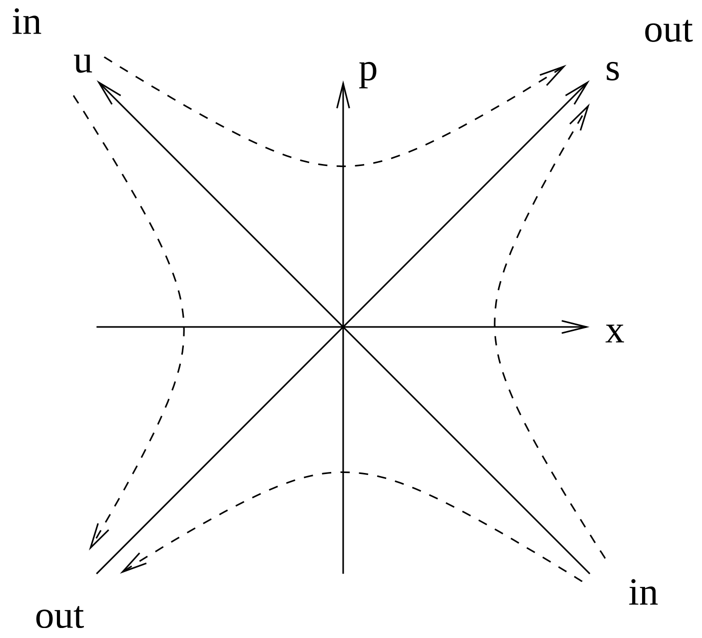

Figure 4: The phase space for a harmonic oscillator parameterized by the x, p coordinates or the s, u coordinates. The dotted lines denote various possible classical trajectories.

operators of the ordinary harmonic oscillator and their eigenstates are analogous to the familiar coherent states. However, unlike the ordinary harmonic oscillator, since $p \pm x$ are *hermitian* operators, their eigenvalues are real and the inner product of functions in these representations is standard.

We will present this formalism, will clarify some of its properties and will extend it. We will start the discussion of the first quantized theories with some general properties of eigenstates of $p \pm x$, and will then use them in the special cases relevant to the 0B and 0A strings. Then, we will study the second quantized theories and will compute their free energies.

A.1 First quantized problems

A single upside down harmonic oscillator

Consider first a generic quantum mechanical problem of a single degree of freedom. Standard bases of orthonormal states are $|x\rangle$ and $|p\rangle$, which are coordinate and momentum eigenstates respectively, with $\langle x|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipx}$. We will also be interested in the bases $|s\rangle$ and $|u\rangle$ which are orthonormal eigenstates of

$$\begin{aligned} S &= \frac{P+X}{\sqrt{2}} \\ U &= \frac{P-X}{\sqrt{2}} \\ [S, U] &= i\,. \end{aligned} \tag{A.1}$$

It is easy to find the inner products

$$
\begin{aligned}
\langle x|s\rangle &= \frac{2^{\frac{1}{4}}e^{i\frac{\pi}{8}}}{\sqrt{2\pi}}\exp\left(i\left(-\frac{x^2}{2}+\sqrt{2}sx-\frac{s^2}{2}\right)\right)\\
\langle x|u\rangle &= \frac{2^{\frac{1}{4}}e^{-i\frac{\pi}{8}}}{\sqrt{2\pi}}\exp\left(i\left(\frac{x^2}{2}+\sqrt{2}ux+\frac{u^2}{2}\right)\right)\\
\langle s|u\rangle &= \frac{1}{\sqrt{2\pi}}\exp\left(isu\right). \qquad \text{(A.2)}
\end{aligned}
$$

The s and u dependent phases in $|s\rangle$ and $|u\rangle$ are such that $U = S-\sqrt{2}X$ acts on $\langle s|u\rangle$, and $\langle s|x\rangle$ as $-i\partial_s$ and similarly for the action of S on $\langle u|s\rangle$ and $\langle u|x\rangle$. In the last expression we defined the integral $\int dx e^{ix^2}$ as $\int dx e^{(i-0^+)x^2} = \sqrt{i\pi}$. We have chosen the constant phases of the first two lines so as to simplify the last line and some of the subsequent formulas.

So, let us now focus on the inverted harmonic oscillator with the lagrangian and hamiltonian

$$
\begin{aligned}
L &= \frac{1}{2}(\dot{X}^2+X^2)\\
H &= \frac{1}{2}(P^2-X^2) = \frac{1}{2}(SU+US). \qquad \text{(A.3)}
\end{aligned}
$$

It is easy to work out the time evolution

$$
\begin{aligned}
e^{-iHt}|s\rangle &= e^{\frac{t}{2}}|e^t s\rangle\\
e^{-iHt}|u\rangle &= e^{-\frac{t}{2}}|e^{-t} u\rangle\\
\langle s|e^{-iHt} &= e^{-\frac{t}{2}}\langle e^{-t} s|\\
\langle u|e^{-iHt} &= e^{\frac{t}{2}}\langle e^{t} u|\\
\langle s|e^{-iHt}|u\rangle &= \frac{1}{\sqrt{2\pi}}e^{-\frac{t}{2}}\exp\left(isue^{-t}\right). \qquad \text{(A.4)}
\end{aligned}
$$

Here the factors of $e^{\pm\frac{t}{2}}$ are needed for unitarity, but also come out of (A.3) by writing, in the s basis $H = -is\partial_s - \frac{i}{2}$, and $H = iu\partial_u + \frac{i}{2}$ in the u basis.

The operators (A.12) are similar to the creation and annihilation operators of the ordinary (or "upside up") harmonic oscillator. One difference is that in our case, S and U are hermitian operators and not hermitian conjugates to each other. Correspondingly the states $|s\rangle$ or $|u\rangle$ are analogous to coherent states. Since these operators are hermitian the states $\langle s|$ and $\langle u|$ will also eigenstates of S and U respectively. These two bases will be useful to describe the initial and final states of the upside down harmonic oscillator. In other words, the incoming states will be naturally described in terms of the u basis and the outgoing states in terms of the s basis. This can be seen quite naturally by looking at the shape of trajectories in 4, but will be seen more precisely later.

There are two linearly independent energy eigenstates for every energy ϵ. In the x representation the wavefunctions are the two parabolic cylinder functions. They can be taken to be even and odd under the parity transformation $x \to -x$. Alternatively, we can take one wavefunction to correspond to a wave coming from the left and scattered to the right and back to the left, and the other wave function obtained from this one by $x \to -x$.

In the s representation the energy eigenstates with eigenvalue ϵ are $s^{i\epsilon-\frac{1}{2}}$. The singularity at $s=0$ leads to a two fold doubling of the number of states $|\epsilon,\text{out}\pm\rangle$, where the label *out* will be explained shortly. Their wavefunctions are

$$\begin{aligned}
\langle s|\epsilon,\text{out}+\rangle &= \begin{cases} \frac{1}{\sqrt{2\pi}} s^{i\epsilon-\frac{1}{2}} & s>0 \\ 0 & s<0 \end{cases} \\
\langle s|\epsilon,\text{out}-\rangle &= \begin{cases} 0 & s>0 \\ \frac{1}{\sqrt{2\pi}} (-s)^{i\epsilon-\frac{1}{2}} & s<0 \end{cases}
\end{aligned} \tag{A.5}$$

By looking at the trajectories in 4 we see that $|\epsilon,\text{out}+\rangle$ states are states that in their outgoing modes contain only a right moving piece. While the states $|\epsilon,\text{out}-\rangle$ contain only a left moving piece in their outgoing modes. Therefore we will refer to them as "*out* states".

Another natural basis arises from the u representation

$$\begin{aligned}
\langle u|\epsilon,\text{in}+\rangle &= \begin{cases} \frac{1}{\sqrt{2\pi}} u^{-i\epsilon-\frac{1}{2}} & u>0 \\ 0 & u<0 \end{cases} \\
\langle u|\epsilon,\text{in}-\rangle &= \begin{cases} 0 & u>0 \\ \frac{1}{\sqrt{2\pi}} (-u)^{-i\epsilon-\frac{1}{2}} & u<0 \end{cases}
\end{aligned} \tag{A.6}$$

These are states which contain only right/left moving incoming pieces for $+/-$. We will refer to them as "*in* states".

Semiclassically the incoming states have $u\to\pm\infty$ and $s\approx 0$, while the outgoing states have $u\approx 0$ and $s\to\pm\infty$. Therefore, it is natural to take the incoming states to be $|\epsilon,\text{in}\pm\rangle$, where $|\epsilon,\text{in}+\rangle$ describes a particle coming from the left (negative x) and $|\epsilon,\text{in}-\rangle$ describes a particle coming from the right (positive x). Similarly, the outgoing states are $|\epsilon,\text{out}\pm\rangle$. Here, $|\epsilon,\text{out}+\rangle$ describes a particle going to the right (positive x), and $|\epsilon,\text{out}-\rangle$ describes a particle going to the left (negative x).

These two bases are related by a unitary transformation

$$\begin{aligned}
\begin{pmatrix} |\epsilon,\text{out}+\rangle \\ |\epsilon,\text{out}-\rangle \end{pmatrix} &= \mathcal{S}\begin{pmatrix} |\epsilon,\text{in}+\rangle \\ |\epsilon,\text{in}-\rangle \end{pmatrix} \\
\mathcal{S} &= \begin{pmatrix} \frac{e^{i\frac{\pi}{4}}e^{\frac{\pi\epsilon}{2}}}{\sqrt{2\pi}}\Gamma(\frac{1}{2}-i\epsilon) & \frac{e^{-i\frac{\pi}{4}}e^{-\frac{\pi\epsilon}{2}}}{\sqrt{2\pi}}\Gamma(\frac{1}{2}-i\epsilon) \\ \frac{e^{-i\frac{\pi}{4}}e^{-\frac{\pi\epsilon}{2}}}{\sqrt{2\pi}}\Gamma(\frac{1}{2}-i\epsilon) & \frac{e^{i\frac{\pi}{4}}e^{\frac{\pi\epsilon}{2}}}{\sqrt{2\pi}}\Gamma(\frac{1}{2}-i\epsilon \end{pmatrix} = \begin{pmatrix} \frac{e^{i\frac{\pi}{4}}e^{i\Phi_B(\epsilon)}}{\sqrt{1+e^{-2\pi\epsilon}}} & \frac{e^{-i\frac{\pi}{4}}e^{i\Phi_B(\epsilon)}}{\sqrt{1+e^{2\pi\epsilon}}} \\ \frac{e^{-i\frac{\pi}{4}}e^{i\Phi_B(\epsilon)}}{\sqrt{1+e^{2\pi\epsilon}}} & \frac{e^{i\frac{\pi}{4}}e^{i\Phi_B(\epsilon)}}{\sqrt{1+e^{-2\pi\epsilon}}} \end{pmatrix} \\
e^{i\Phi_B(\epsilon)} &= \sqrt{\frac{\Gamma(\frac{1}{2}-i\epsilon)}{\Gamma(\frac{1}{2}+i\epsilon)}}.
\end{aligned} \tag{A.7}$$

Here we wrote $\langle\epsilon,\text{out}+|\epsilon,in-\rangle = \int dsdu\langle\epsilon,\text{out}+|s\rangle\langle s|u\rangle\langle u|\epsilon,\text{in}-\rangle$ and we used (A.5) , (A.6) , (A.2) . Another way to understand these bases is to express the *out* states as functions of u and the *in* states as functions of s. Using (A.7), or more directly by using $\langle s|u\rangle$ in (A.2) and Fourier transforming (A.5)(A.6), we find

$$\langle s|\epsilon,\text{in}+\rangle = \frac{e^{i\frac{\pi}{4}}e^{i\Phi_B(\epsilon)}}{\sqrt{2\pi}\sqrt{1+e^{-2\pi\epsilon}}}(s+i0^+)^{i\epsilon-\frac{1}{2}}$$

$$
\begin{aligned}
\langle s|\epsilon,\mathrm{in}-\rangle &= \frac{e^{-i\frac{\pi}{4}}e^{i\Phi_B(\epsilon)}}{\sqrt{2\pi}\sqrt{1+e^{2\pi\epsilon}}}(s+i0^-)^{i\epsilon-\frac{1}{2}} \\
\langle u|\epsilon,\mathrm{out}+\rangle &= \frac{e^{-i\frac{\pi}{4}}e^{-i\Phi_B(\epsilon)}}{\sqrt{2\pi}\sqrt{1+e^{-2\pi\epsilon}}}(u+i0^-)^{-i\epsilon-\frac{1}{2}} \\
\langle u|\epsilon,\mathrm{out}-\rangle &= \frac{e^{i\frac{\pi}{4}}e^{-i\Phi_B(\epsilon)}}{\sqrt{2\pi}\sqrt{1+e^{2\pi\epsilon}}}(u+i0^+)^{-i\epsilon-\frac{1}{2}}
\end{aligned}
\tag{A.8}
$$

where the $s+i0^{\pm}$ prescription means that for negative s we substitute $s^{i\alpha} = (|s|e^{\pm i\pi})^{i\alpha} = (-s)^{i\alpha}e^{\mp\pi\alpha}$. So the $|\epsilon, in\pm\rangle$ states in the s representation $\langle s|\epsilon, in\pm\rangle$ are linear combinations of $\langle s|\epsilon,\mathrm{out}\pm\rangle$ whose precise coefficients are determined by thinking of $\langle s|\epsilon,\mathrm{in}\pm\rangle$ as functions that are analytic in the upper/lower half s plane. The situation is very similar to one that arises in the physics of Rindler space when we express the Minkowski wavefunctions in terms of the Rindler wavefunctions. In our case this arises from the fact that $|\epsilon,\mathrm{in}+\rangle$ has support only for $u>0$, this implies that in the s representation this is an analytic function for $Im(s)>0$. In the Rindler case, the positive frequency condition on the Minkowski wavefunctions implies similar analyticity properties. In [48] a connection between these fermions and fermions in de-Sitter space was studied. In [49] the thermal looking nature of these amplitudes was explored.

The even and odd states $|\epsilon,\mathrm{out}+\rangle \pm |\epsilon,\mathrm{out}-\rangle$ and $|\epsilon,\mathrm{in}+\rangle \pm |\epsilon,\mathrm{in}-\rangle$ diagonalize (A.7)

$$
\begin{aligned}
|\epsilon,\mathrm{in}+\rangle \pm |\epsilon,\mathrm{in}-\rangle &= e^{i\varphi_\pm(\epsilon)}\left(|\epsilon,\mathrm{out}+\rangle \pm |\epsilon,\mathrm{out}-\rangle\right) \\
e^{i\varphi_+(\epsilon)} &= e^{i\Phi_B(\epsilon)}\frac{e^{i\frac{\pi}{4}}+e^{-i\frac{\pi}{4}}e^{-\pi\epsilon}}{\sqrt{1+e^{-2\pi\epsilon}}} = 2^{-i\epsilon}\frac{\Gamma(\frac{1}{4}-\frac{i\epsilon}{2})}{\Gamma(\frac{1}{4}+\frac{i\epsilon}{2})} \\
e^{i\varphi_-(\epsilon)} &= e^{i\Phi_B(\epsilon)}\frac{e^{i\frac{\pi}{4}}-e^{-i\frac{\pi}{4}}e^{-\pi\epsilon}}{\sqrt{1+e^{-2\pi\epsilon}}} = 2^{-i\epsilon}i\frac{\Gamma(\frac{3}{4}-\frac{i\epsilon}{2})}{\Gamma(\frac{3}{4}+\frac{i\epsilon}{2})}\,.
\end{aligned}
\tag{A.9}
$$

Our interpretation of (A.6) (A.5) as *in* and *out* states is further supported by comparing (A.5)(A.6) and (A.8). The *in* states $|\epsilon,\mathrm{in}\pm\rangle$ have support only for one sign of u and for both signs of s. This is the expected behavior of incoming states. The *out* states $|\epsilon,\mathrm{out}\pm\rangle$ have support only for one sign of s and for both signs of u. This is the expected behavior of outgoing states. Furthermore, for large $|\epsilon|$ the relation between the two bases is simple. Up to an ϵ dependent phase $|\epsilon,\mathrm{in}\pm\rangle \sim |\epsilon,\mathrm{out}\pm\rangle$ for positive ϵ and $|\epsilon,\mathrm{in}\pm\rangle \sim |\epsilon,\mathrm{out}\mp\rangle$ for negative ϵ. This is consistent with the semiclassical picture of complete transmission for positive ϵ and complete reflection for negative ϵ.

One can actually show more precisely why it is reasonable associate the basis $|\epsilon,\mathrm{in}\pm\rangle$ with incoming states. For that purpose we can compute $\langle x|\epsilon,\mathrm{in}\pm\rangle$ using (A.2) (A.6) . We do not need the exact answer, which is a combination of parabolic cylinder functions. We only need the behavior of the function for large x. Since x is large we can compute the answer by saddle point integration. The saddle point equation for u is $\sqrt{2}x+u-\epsilon/u\sim 0$. The two saddle points, at $u\sim-\sqrt{2}x$ and at $u\sim\epsilon/(\sqrt{2}x)$, give the incoming and outgoing pieces of the x space wavefunction, which go like $e^{-i\frac{x^2}{2}}$ and $e^{i\frac{x^2}{2}}$ to leading order in x. Note that the first saddle point arises only for $\pm x<0$ for $|\epsilon, in\pm\rangle$. This means that the incoming wavefunction is supported to the left/right side of the potential for $|\epsilon, in\pm\rangle$. Furthermore, the coefficient of the first saddle point is energy independent (except for a simple, expected, factor of $|x|^{-i\epsilon-\frac{1}{2}}$). This is the natural normalization for the incoming states. The integral in the region close to the second saddle point gives us the reflected part of the wavefunction and contains the information about scattering phase.

Repeating this discussion for $|\epsilon, \mathrm{out}\pm\rangle$ we can understand why it is natural to associate them to outgoing states which are right or left moving.

Two upside down harmonic oscillators

Now, let us discuss the same problem but with two degrees of freedom X_1 and X_2 and their conjugate momenta P_1 and P_2. We change variables to polar coordinates $X_1 = X\cos\theta$, $X_2 = X\sin\theta$. The momentum conjugate to θ is

$$q = X_1P_2 - X_2P_1 \tag{A.10}$$

and we can work in a sector where it is a fixed integer c number. Then, we have two natural bases of states $|x\rangle$ and $|p\rangle$ which are eigenstates of X and $P = \cos\theta P_1 + \sin\theta P_2$ respectively. Note that, even though (X_i, P_i) are canonically conjugate, P is not the momentum conjugate to X.

As in (A.12), we define

$$\begin{aligned} S_i &= \frac{P_i + X_i}{\sqrt{2}} \\ U_i &= \frac{P_i - X_i}{\sqrt{2}} \\ [S_i, U_j] &= i\delta_{i,j} \end{aligned} \tag{A.11}$$

and we can again change to "polar coordinates"

$$\begin{aligned} S_1 &= S\cos\theta_s \\ S_2 &= S\sin\theta_s \\ U_1 &= U\cos\theta_u \\ U_2 &= U\sin\theta_u\,. \end{aligned} \tag{A.12}$$

The momenta conjugate to S and U are P_s and P_u. It is important that they are not given by U and $-S$. However, q of (A.10) can be written also as $q = S_1U_2 - S_2U_1$, and it is the momentum conjugate to both θ, θ_s and θ_u. This follows from the fact that this is the charge of the same rotation symmetry.

Let us study the various bases in more detail. The simplest states are X_i eigenstates, $|x_1, x_2\rangle$, or in polar coordinates $|x, \theta\rangle$. Note that the eigenvalue x is positive. They satisfy

$$\langle x_1, x_2|x, \theta\rangle = \sqrt{x}\delta(x_1 - x\cos\theta)\delta(x_2 - x\sin\theta)\,. \tag{A.13}$$

Instead of diagonalizing θ it is better to diagonalize q, $|x, q\rangle = \frac{1}{\sqrt{2\pi}}\int d\theta e^{iq\theta}|x, \theta\rangle$. We will use similar notation for various bases diagonalizing the S or U variables (again, the eigenvalues s and u are positive). One way to find the inner products between these bases

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is to convert to the bases where the cartesian coordinates X_i, S_i or U_i are diagonal and then use (A.2) for each of them. We readily find (recall, $\int_0^{2\pi} d\theta e^{iq\theta} e^{ia\cos\theta} = 2\pi i^q J_q(a)$)

$$\begin{aligned}
\langle s,\theta_s|x,q\rangle &= \frac{e^{-i\frac{\pi}{4}} i^{\frac{-3q}{2}} \sqrt{2xs}}{(2\pi)^{\frac{3}{2}}} \int_0^{2\pi} d\theta e^{iq\theta} \exp\left(i\left(\frac{x^2}{2} - \sqrt{2}sx\cos(\theta-\theta_s) + \frac{s^2}{2}\right)\right) \\
&= \frac{e^{-i\frac{\pi}{4}}(-1)^q i^{-\frac{q}{2}} \sqrt{2xs}}{\sqrt{2\pi}} \exp\left(iq\theta_s + i\frac{x^2}{2} + i\frac{s^2}{2}\right) J_q(\sqrt{2}sx) \\
\langle u,\theta_u|x,q\rangle &= \frac{e^{i\frac{\pi}{4}} i^{-\frac{q}{2}} \sqrt{2xu}}{(2\pi)^{\frac{3}{2}}} \int_0^{2\pi} d\theta e^{iq\theta} \exp\left(i\left(-\frac{x^2}{2} - \sqrt{2}ux\cos(\theta-\theta_u) - \frac{u^2}{2}\right)\right) \\
&= \frac{e^{i\frac{\pi}{4}}(-1)^q i^{\frac{q}{2}} \sqrt{2xu}}{\sqrt{2\pi}} \exp\left(iq\theta_u - i\frac{x^2}{2} - i\frac{u^2}{2}\right) J_q(\sqrt{2}ux) \\
\langle s,q|u,q'\rangle &= \frac{i^{-q}\delta_{q,q'}\sqrt{su}}{2\pi} \int_0^{2\pi} d\theta_u e^{iq\theta_u + isu\cos\theta_u} = \sqrt{su} J_q(su)\delta_{q,q'}
\end{aligned} \tag{A.14}$$

where we have chosen the overall phases to simplify some formulas later. The first two expressions demonstrate that q is the momentum conjugate to θ, θ_s and θ_u. The last inner product can be derived in several different ways whose consistency relies on (or better, gives a proof of) the Weber's formula

$$\int_0^\infty e^{-px^2} J_q(ax) J_q(bx) x dx = \frac{e^{-\frac{a^2+b^2}{4p}}}{2p} I_q\left(\frac{ab}{2p}\right) = \frac{e^{-\frac{a^2+b^2}{4p}}}{2p} i^{-q} J_q\left(\frac{iab}{2p}\right) \tag{A.15}$$

with $p = -i + 0^+$.

We now study the system with the hamiltonian

$$\begin{aligned}
H &= \frac{1}{2}(P_1^2 + P_2^2 - X_1^2 - X_2^2) = \frac{1}{2}\left(P^2 + \frac{q^2 - \frac{1}{4}}{X^2} - X^2\right) \\
&= \frac{1}{2}(S_1U_1 + U_1S_1 + S_2U_2 + U_2S_2) \\
&= \frac{1}{2}(SP_s + P_sS) \\
&= -\frac{1}{2}(UP_u + P_uU)\,.
\end{aligned} \tag{A.16}$$

We will take q to be a c number and will view the system as having a single degree of freedom. This hamiltonian has two natural energy eignestates $|\epsilon,\mathrm{in}\rangle$ and $|\epsilon,out\rangle$ with wavefunctions and inner products

$$\begin{aligned}
\langle s|\epsilon,out\rangle &= \frac{1}{\sqrt{2\pi}} s^{i\epsilon-\frac{1}{2}} \\
\langle u|\epsilon,in\rangle &= \frac{1}{\sqrt{2\pi}} u^{-i\epsilon-\frac{1}{2}} \\
\langle \epsilon,out|\epsilon',in\rangle &= e^{i\Phi_A(\epsilon)}\delta(\epsilon-\epsilon') = 2^{-i\epsilon}\frac{\Gamma\left(\frac{1}{2}(1+q-i\epsilon)\right)}{\Gamma\left(\frac{1}{2}(1+q+i\epsilon)\right)}\delta(\epsilon-\epsilon')
\end{aligned} \tag{A.17}$$

where in the last inner product we used the integral $\int_0^\infty dy y^{-i\epsilon} J_q(y) = 2^{-i\epsilon}\frac{\Gamma\left(\frac{1}{2}(1+q-i\epsilon)\right)}{\Gamma\left(\frac{1}{2}(1+q+i\epsilon)\right)}$ with nonnegative q. Note, as a check that these inner products are independent of the sign of q.

A.2 Second quantized problem

0B

The 0B matrix model is a system of fermions whose first quantized description is the first problem discussed above. Its lagrangian is

$$\mathcal{L} = \int_{-\infty}^{\infty} dx \Psi^{\dagger}(x,t)\left(i\partial_t + \frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 + \mu\right)\Psi(x,t)\,. \tag{A.18}$$

In order to express it in terms of the s and u variables we define the fermionic fields

$$\begin{aligned}
\Psi_s(s,t) &= \int dx\langle s|x\rangle\Psi(x,t) = \frac{2^{\frac{1}{4}}e^{-i\frac{\pi}{8}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx \exp\left(i\left(\frac{x^2}{2} - \sqrt{2}sx + \frac{s^2}{2}\right)\right)\Psi(x,t) \\
\Psi_u(u,t) &= \int dx\langle u|x\rangle\Psi(x,t) = \frac{2^{\frac{1}{4}}e^{i\frac{\pi}{8}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx \exp\left(i\left(-\frac{x^2}{2} - \sqrt{2}ux - \frac{u^2}{2}\right)\right)\Psi(x,t)
\end{aligned} \tag{A.19}$$

which are related through a Fourier transform

$$\Psi_s(s,t) = \int du\langle s|u\rangle\Psi_u(u,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} du \exp\left(isu\right)\Psi_u(u,t) \tag{A.20}$$

and find

$$\begin{aligned}
\mathcal{L} &= \int_{-\infty}^{\infty} ds \Psi_s^{\dagger}(s,t)\left(i\partial_t + \frac{i}{2}(s\partial_s + \partial_s s) + \mu\right)\Psi_s(s,t) \\
&= \int_{-\infty}^{\infty} du \Psi_u^{\dagger}(u,t)\left(i\partial_t - \frac{i}{2}(u\partial_u + \partial_u u) + \mu\right)\Psi_u(u,t)\,.
\end{aligned} \tag{A.21}$$

We can further simplify the analysis by the change of variables

$$\begin{aligned}
\Psi_1^{(in)}(r,t) &= e^{\frac{r}{2}}\Psi_u(u = e^r, t) \\
\Psi_2^{(in)}(r,t) &= e^{\frac{r}{2}}\Psi_u(u = -e^r, t) \\
\Psi_1^{(out)}(r,t) &= e^{\frac{r}{2}}\Psi_s(s = e^r, t) \\
\Psi_2^{(out)}(r,t) &= e^{\frac{r}{2}}\Psi_s(s = -e^r, t)
\end{aligned} \tag{A.22}$$

which makes the lagrangians (A.21) look relativistic

$$\begin{aligned}
\mathcal{L} &= \int_{-\infty}^{\infty} dr \sum_{i=1,2}\Psi_i^{(in)\dagger}(r,t)\left(i\partial_t - i\partial_r + \mu\right)\Psi_i^{(in)}(r,t) \\
&= \int_{-\infty}^{\infty} dr \sum_{i=1,2}\Psi_i^{(out)\dagger}(r,t)\left(i\partial_t + i\partial_r + \mu\right)\Psi_i^{(out)}(r,t)\,.
\end{aligned} \tag{A.23}$$

The parameter μ is like the time component of a vector field coupled to the fermion number current whose incoming and outgoing components are

$$\begin{aligned}
J^{(in)} &= \sum_i \Psi_i^{(in)\dagger}\Psi_i^{(in)} \\
J^{(out)} &= \sum_i \Psi_i^{(out)\dagger}\Psi_i^{(out)}\,.
\end{aligned} \tag{A.24}$$

We can remove it by a time dependent gauge transformation, but we prefer not to do so. In this form it is clear that we have four incoming Majorana Weyl fermions $\Psi^{(in)}$ and four outgoing Majorana Weyl fermions $\Psi^{(out)}$ of the opposite chirality. The incoming and the outgoing fermions are related through our map (A.20) which becomes

$$\begin{aligned}\Psi_1^{(out)}(r,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dr' e^{\frac{1}{2}(r+r')}\left(\exp(ie^{r+r'})\Psi_1^{(in)}(r',t)+\exp(-ie^{r+r'})\Psi_2^{(in)}(r',t)\right)\\ \Psi_2^{(out)}(r,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dr' e^{\frac{1}{2}(r+r')}\left(\exp(-ie^{r+r'})\Psi_1^{(in)}(r',t)+\exp(ie^{r+r'})\Psi_2^{(in)}(r',t)\right).\end{aligned} \tag{A.25}$$

What are the symmetries of our problem? In the form (A.23) the lagrangian has an incoming SO(4) $\approx$ SU(2) $\times$ SU(2)$'$ symmetry which rotates the incoming fermions and similarly an outgoing SO(4) symmetry which rotates the outgoing fermions. The coupling to μ breaks each of these symmetries to SU(2)$\times$U(1). The map between the incoming and outgoing fields (A.25) breaks most of these symmetries. But, let us first ignore the map and start, without loss of generality, by considering the incoming symmetry. The current of the U(1) factor has already been mentioned in (A.24). The incoming SU(2) currents are $J^{(in)+} = \Psi_1^{(in)\dagger}\Psi_2^{(in)}$, $J^{(in)-} = \Psi_2^{(in)\dagger}\Psi_1^{(in)}$ and $J^{(in)0} = \Psi_1^{(in)\dagger}\Psi_1^{(in)} - \Psi_2^{(in)\dagger}\Psi_2^{(in)}$. We note that the currents $J^{(in)}$ of (A.24) and $J^{(in)0}$ are local in our original "space" coordinate s, while the currents $J^{(in)\pm}$ are nonlocal. The latter involve creating a fermions at s and annihilating a fermion at $-s$, or the other way around. The same distinction between these currents applies in the x coordinate.

These four currents have an obvious string theory interpretation. After bosonization the fermion number current $J^{(in)}$ creates an incoming NS-NS tachyon $T^{(in)}$; roughly[13] $J^{(in)} \sim (\partial_t - \partial_\phi)T^{(in)}$. The current $J^{(in)0}$ creates the incoming R-R scalar $C^{(in)}$; roughly $J^{(in)0} \sim (\partial_t - \partial_\phi)C^{(in)}$. These two excitations, which are local in x, correspond to the perturbative string spectrum. The other two currents $J^{(in)\pm}$ create nonperturbative string states. These are solitons – coherent states of an infinite number of $C^{(in)}$ quanta; roughly $J^{(in)\pm} \sim e^{\pm i\sqrt{2}C^{(in)}}$. Such solitons were studied in [6] . They create a fermion at one sign of x and annihilate it at the other. Note that this is consistent with the C field being at the SU(2) radius. We now interpret this SU(2) symmetry as rotating the two fermion flavors in (A.23).

This discussion of the incoming symmetries is trivially repeated for the outgoing symmetries. In terms of the field C the SU(2) symmetries of the past and the future are simply those of the left moving and the right moving fields at the selfdual radius.

The map from the past to the future (A.25) shows that only one of the two U(1) symmetries is conserved — the total incoming fermion number equals the total outgoing fermion number. The two SU(2) symmetries are more interesting. Both of them are broken, but for large $|\mu|$ a certain U(1) $\subset$ SU(2)$^{(in)}$ $\times$ SU(2)$^{(out)}$ is approximately conserved. It is broken only by nonperturbative effects of order $e^{-c|\mu|}$ for some constant c. The physical interpretation of this fact is simple. We start with a vacuum with $N \to \infty$ fermions. Let us

[13] We use the word "roughly" because of the non-local transform between ϕ and r.

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prepare an initial state with $\frac{1}{2}N+n^{(in)}$ incoming fermions from negative x and $\frac{1}{2}N-n^{(in)}$ incoming fermions from positive x, and let us examine a final state with $\frac{1}{2}N+n^{(out)}$ outgoing fermions to positive x and $\frac{1}{2}N-n^{(out)}$ outgoing fermions to negative x. For $\mu\to-\infty$, where the fermions are far below the barrier, there is almost no communication between the left and right sides of the potential, and we must have $n^{(in)}\approx-n^{(out)}$. Conversely, for $\mu\to+\infty$ we have $n^{(in)}\approx n^{(out)}$. But for finite μ we can have arbitrary $n^{(in)}$ and $n^{(out)}$. Such scattering processes are created with the insertion of $n^{(in)}$ insertions of $e^{i\sqrt{2}C^{(in)}}$ in the past (for negative $n^{(in)}$ we take $e^{-i\sqrt{2}C^{(in)}}$), and $n^{(out)}$ insertions of $e^{i\sqrt{2}C^{(out)}}$ in the future. The condition $n^{(in)}\approx n^{(out)}$ for $\mu\to+\infty$ states that the winding of C is approximately conserved while the momentum of C is not conserved. For $\mu\to-\infty$ we have the reverse situation.

Let us discuss the discrete symmetries of our problem. First, a $\mathbf{Z}_2$ subgroup of the the SU(2) we mentioned above is not broken by the map (A.25). Combining it with a U(1) transformation we can identify it as the parity transformation $x\to-x$, $s\to-s$ and $u\to-u$. In terms of the in and out fermions its action is $\Psi_1^{(in/out)}(r,t)\leftrightarrow\Psi_2^{(in/out)}(r,t)$. We identify this transformation with the spacetime charge conjugation which is generated by the worldsheet transformation $(-1)^{F_L}$ (F_L is the leftmoving spacetime fermion number). As a check, note that the currents $J^{(in/out)}$ are even and the currents $J^{(in/out)\pm}$ are odd under this transformation. Note that spacetime charge conjugation is parity in the matrix model.

Let us consider now the S-duality symmetry in spacetime, which is generated in the worldsheet description by $(-1)^{f_L}$ (f_L is the leftmoving spacetime fermion number). It acts on the parameter μ as $\mu\to-\mu$, and therefore it is a symmetry only for $\mu=0$. Its action on the fields is

$$\begin{aligned}\Psi(x,t)&\to\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dx'e^{ix'x}\Psi^\dagger(x',t)\\ \Psi_s(s,t)&\to\Psi_s^\dagger(s,t)\\ \Psi_u(u,t)&\to\Psi_u^\dagger(-u,t)\\ \Psi_1^{(in)}&\leftrightarrow\Psi_2^{(in)\dagger}\\ \Psi_i^{(out)}&\to\Psi_i^{(out)\dagger}\,.\end{aligned}\tag{A.26}$$

It is easy to check that it is a symmetry of the lagrangian (A.18)(A.21)(A.23) with $\mu=0$, and of the transforms (A.19)(A.20)(A.25). Clearly, if we combine this operation with the parity transformation $(-1)^{F_L}$ the difference between the transformations of Ψ_s and Ψ_u and the difference between $\Psi^{(in)}$ and $\Psi^{(out)}$ are reversed. It is interesting that while in x space the transformation involves a duality transformation of (x,p), which is implemented by a Fourier transform, in s, u and r space no such duality transformation is needed, and the transformation rules are local and simple. We conclude that in the s, u and r variables this $\mathbf{Z}_2$ symmetry acts as charge conjugation on the outgoing fields and as CP on the ingoing fields.

We point out that this $\mathbf{Z}_2$ symmetry is a subgroup of the original SO(4) we mentioned above.

0A

The 0A matrix model is a system of fermions whose first quantized description is the second problem discussed above. Its lagrangian is

$$\mathcal{L} = \int_0^\infty dx \Psi^\dagger(x,t) \left(i\partial_t + \frac{1}{2}\left(\partial_x^2 + x^2 - \frac{q^2 - \frac{1}{4}}{x^2} \right) + \mu \right) \Psi(x,t) \,. \tag{A.27}$$

In order to express it in terms of the s and u variables we define new fermionic fields which are related by integral transforms

$$\begin{aligned}
\Psi_s(s,t) &= \int dx \langle s,q|x,q\rangle \Psi(x,t) = e^{-i\pi(\frac{1}{4}-\frac{3q}{4})} \int_0^\infty dx \sqrt{2xs}\, e^{\frac{i}{2}(x^2+s^2)} J_q(\sqrt{2}sx)\Psi(x,t) \\
\Psi_u(u,t) &= \int dx \langle u,q|x,q\rangle \Psi(x,t) = e^{i\pi(\frac{1}{4}-\frac{3q}{4})} \int_0^\infty dx \sqrt{2xu} e^{-\frac{i}{2}(x^2+u^2)} J_q(\sqrt{2}ux)\Psi(x,t) \\
\Psi_s(s,t) &= \int du \langle s,q|u,q\rangle \Psi_u(u,t) = \int_0^\infty du \sqrt{su} J_q(su) \Psi_u(u,t) \,.
\end{aligned} \tag{A.28}$$

As in (A.22) it is convenient to express them in terms of incoming and outgoing fermions

$$\begin{aligned}
\Psi^{(in)}(r,t) &= e^{\frac{r}{2}} \Psi_u(u = e^r) \\
\Psi^{(out)}(r,t) &= e^{\frac{r}{2}} \Psi_s(s = e^r) \\
\Psi^{(out)}(r,t) &= \int_{-\infty}^\infty dr' e^{r+r'} J_q(e^{r+r'}) \Psi^{(in)}(r',t) \,.
\end{aligned} \tag{A.29}$$

It is easy to express the lagrangian (A.27) in these variables

$$\begin{aligned}
\mathcal{L} &= \int_0^\infty ds \Psi_s^\dagger(s,t) \left(i\partial_t + \frac{i}{2}(s\partial_s + \partial_s s) + \mu \right) \Psi_s(s,t) \\
&= \int_0^\infty du \Psi_u^\dagger(u,t) \left(i\partial_t - \frac{i}{2}(u\partial_u + \partial_u u) + \mu \right) \Psi_u(u,t) \\
&= \int_0^\infty dr \Psi^{(in)\dagger}(r,t) \left(i\partial_t - i\partial_r + \mu \right) \Psi^{(in)}(r,t) \\
&= \int_0^\infty dr \Psi^{(out)\dagger}(r,t) \left(i\partial_t + i\partial_r + \mu \right) \Psi^{(out)}(r,t) \,.
\end{aligned} \tag{A.30}$$

As in the 0B theory we have a U(1) symmetry which rotates Ψ by a phase and corresponds to fermion number conservation. The spacetime charge conjugation symmetry, which is generated by $(-1)^{F_L}$ on the worldsheet, acts as $q \to -q$. Up to an overall phase this acts leaving $\Psi^{(in)}$ invariant and changing $\Psi^{(out)} \to (-1)^q \Psi^{(out)}$, which is somewhat trivial. The S-duality transformation, $(-1)^{f_L}$, acts on the parameter μ as $\mu \to -\mu$, and is a symmetry only for $\mu = 0$. Its action on the fields is

$$\begin{aligned}
\Psi(x,t) &\to \int_0^\infty dx' \sqrt{xx'} J_q(xx') \Psi^\dagger(x',t) \\
\Psi_s(s,t) &\to \Psi_s^\dagger(s,t) \\
\Psi_u(u,t) &\to \Psi_u^\dagger(u,t) \\
\Psi^{(in)}(r,t) &\to \Psi^{(in)\dagger}(r,t) \\
\Psi^{(out)}(r,t) &\to \Psi^{(out)\dagger}(r,t) \,.
\end{aligned} \tag{A.31}$$

It is easy to check that it is a symmetry of the lagrangian (A.27)(A.30) with $\mu = 0$, and of the transforms (A.28). It is interesting that while in x space the transformation involves an integral transform, in s and u space the transformation is local and it is very simple. We conclude that the S-duality symmetry acts as charge conjugation of the matrix model fermions in the s, u and r variables.

A.3 Computation of the free energies

We are interested in computing the partition function of the thermal system. Using the density of states $\rho(\epsilon) = \frac{\phi'(\epsilon)}{2\pi}$ we can write the standard expression is

$$\begin{aligned}\log \mathcal{Z} &= \lim_{\Lambda\to\infty} \frac{1}{2\pi}\int_{-\Lambda}^{\infty} d\epsilon \phi'(\epsilon)\log(1+e^{-2\pi R(\epsilon-\mu)}) \\ &= \lim_{\Lambda\to\infty} R\left(-\phi(-\Lambda)(\Lambda+\mu) + \int_{-\Lambda}^{\infty} d\epsilon \frac{\phi(\epsilon)}{1+e^{2\pi R(\epsilon-\mu)}}\right) \end{aligned} \tag{A.32}$$

where $-\Lambda$ is a cutoff on the bottom of the Fermi sea and we neglected terms which are exponentially small at large Λ. In order to simplify the analysis and not worrying about the Λ dependence, we will study the second derivative of (A.32)

$$\partial_\mu^2 \log \mathcal{Z} = R\int_{-\infty}^{\infty} d\epsilon \partial_\mu^2 \frac{\phi(\epsilon)}{1+e^{2\pi R(\epsilon-\mu)}} \tag{A.33}$$

which is a convergent integral.

Let us now consider the 0B theory. The determinant of the single particle S-matrix of the 0B theory is $ie^{i\Phi_B(\epsilon)}$. It can be expressed as

$$\Phi_B(\epsilon) = -i\log\left(\frac{\Gamma(\frac{1}{2}-i\epsilon)}{\Gamma(\frac{1}{2}+i\epsilon)}\right) = \int_0^{\infty}\frac{dt}{t}\left(\frac{\sin(\epsilon t)}{\sinh\frac{t}{2}} - 2\epsilon e^{-t}\right) \tag{A.34}$$

where we have used (4.7) . Then, (A.33) becomes

$$\begin{aligned}\partial_\mu^2 \log \mathcal{Z}_B &= R\int_0^{\infty}\frac{dt}{t}\int_{-\infty}^{\infty} d\epsilon \partial_\mu^2 \frac{1}{1+e^{2\pi R(\epsilon-\mu)}}\left(\frac{\sin(\epsilon t)}{\sinh\frac{t}{2}} - 2\epsilon e^{-t}\right) \\ &= -\int_0^{\infty}\frac{dt}{t}\partial_\mu^2\left(\frac{\cos(\mu t)}{2\sinh\frac{t}{2}\sinh\frac{t}{2R}} + R\mu^2 e^{-t}\right). \end{aligned} \tag{A.35}$$

We now want to integrate this equation twice with respect to μ. Invariance under $\mu \to -\mu$ forbids a term linear in μ and the constant term is fixed arbitrarily such that

$$\begin{aligned}\log \mathcal{Z}_B &= -\int_0^{\infty}\frac{dt}{t}\left(\frac{\cos(\mu t)}{2\sinh\frac{t}{2}\sinh\frac{t}{2R}} - \frac{2R}{t^2} + \left[\frac{1}{12}(R+\frac{1}{R}) + R\mu^2\right]e^{-t}\right) \\ &= \Omega(y = 2i\mu, R) + \Omega(\overline{y} = -2i\mu, R)\,. \end{aligned} \tag{A.36}$$

The single particle S-matrix in the 0A theory with nonzero q is $e^{i\Phi_A(\epsilon)}$. It can be expressed as

$$\Phi_A(\epsilon) = -i\log\left(\frac{\Gamma(\frac{1}{2}+\frac{1}{2}(q-i\epsilon))}{\Gamma(\frac{1}{2}+\frac{1}{2}(q+i\epsilon))}\right) = \int_0^{\infty}\frac{dt}{t}\left(\frac{e^{-qt/2}\sin(\frac{\epsilon\, t}{2})}{\sinh\frac{t}{2}} - \epsilon e^{-t}\right) \tag{A.37}$$

where we have used (4.7) , and we dropped a constant term in the phase, as well as a term that is linear in ϵ. The term linear in ϵ could be removed by doing a rescaling of the variables u and s that appeared in the 0A discussion.[14] Then, (A.33) becomes

$$\begin{aligned}\partial_\mu^2 \log \mathcal{Z}_A &= R\int_0^\infty \frac{dt}{t}\int_{-\infty}^\infty d\epsilon\partial_\mu^2 \frac{1}{1+e^{2\pi R(\epsilon-\mu)}}\left(\frac{e^{-qt/2}\sin(\frac{\epsilon\ t}{2})}{\sinh\frac{t}{2}} - \epsilon e^{-t}\right) \\ &= -\int_0^\infty \frac{dt}{t}\partial_\mu^2\left(\frac{e^{-qt/2}\cos(\frac{\mu t}{2})}{2\sinh\frac{t}{2}\sinh\frac{t}{4R}} + \frac{R\mu^2}{2}e^{-t}\right) \\ &= \partial_\mu^2\left[\Omega(y=q+i\mu,2R)+\Omega(\overline{y}=q-i\mu,2R)\right]. \end{aligned} \tag{A.38}$$

Using the definition of the function $\Omega(y,r)$ (3.11) one can derive

$$\Omega(yr,\frac{1}{r}) = \Omega(y,r) - \left[\frac{1}{24}(r+\frac{1}{r}) - \frac{ry^2}{8}\right]\log r\,. \tag{A.39}$$

Using this relation we can check that the 0B answer (A.35) is the same as the 0A answer (A.38) for $q=0$, up to a term involving $\log r$. This term arises because in 0A and 0B it is natural to choose the cutoffs to be R independent. On the other hand T-duality relates them by $\Lambda_B = \Lambda_A\frac{R_A}{\sqrt{2\alpha'}}$ (see (1.2)). Once we take this into account, the terms that are logarithmic in the cutoff give a contribution cancelling the last term in (A.39).

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